The Dynamics of the Outer Satellites of Saturn

by

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Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by David Harper

September 1987

For my dear friends Ruth, Ian and Richard whose kindness and encouragement during the last three years has been a source of inspiration.

Acknowledgements

My warm and sincere thanks to Dr P. J. Message for his invaluable guidance and advice and for many interesting discussions throughout the course of this work.

Thanks also to Dr Andrew Sinclair and Dr Don Taylor of H.M. Nautical Almanac Office at the Royal Greenwich Observatory for giving me so much useful advice and help and for providing me with observational data and computer programs. Without their generosity, much of this work could not have been done.

I would also like to thank Dr G. A. Wilkins and the staff of H.M. Nautical Almanac Office for their kindness and help during my work at the Royal Greenwich Observatory in Summer 1985. It has been a pleasure and a great privilege to be associated with the Observatory.

Finally, thanks to Tracy Colgan, Chris Willis, Sue Sherrell and Andrew Keeley for three very enjoyable years working together.

Special thanks to Tracy Colgan for proof-reading this thesis.

The numerical integration program 'TITAN' described in Chapter 5 was written by Dr. A.T. Sinclair. The description of the program given in section 5.2 is derived from Sinclair and Taylor (1985) and from Dr Sinclair's own notes. The version of the program used in this work is essentially the original version provided by Dr Sinclair, with several minor alterations to enable it to run correctly on the IBM VM/370 system of the University of Liverpool Computer Laboratory.

The observations of the satellites of Saturn used in this work were collected by Dr. D.B. Taylor and punched onto 80-column cards by Mrs. D.E. Oliver at the Royal Greenwich Observatory. As supplied by Dr Taylor, the data files were direct transcripts of the original references, the data being tabulated in several formats according to the format of the source reference. These data files were the raw material for the preparation whose theory is given in Chapter 4. In practise, the preparation involved the writing of several programs to re-format the data and to add auxiliary quantities. The final result is a set of files in a single format which incorporate the original data files plus all auxiliary quantities required for the analysis of the data. This work was carried out at the Royal Greenwich Observatory during the summer of 1985.

The first two figures in Chapter 2 are taken directly from Sinclair (1974) with the permission of Dr Sinclair.

This work was carried out under a research studentship of the Science and Engineering Research Council.

The financial assistance of the University of Liverpool is gratefully acknowledged.

To [celestial mechanics] I was especially attracted because its preparation seemed to me to embody the highest intellectual power to which man had ever attained. The matter used to present itself to my mind somewhat in this way : Supply any man with the fundamental data of astronomy, the times at which stars and planets cross the meridian of a place, and other matters of this kind. He is informed that each of these bodies whose observations he is to use is attracted by all the others with a force which varies as the inverse square of their distances apart. From these data he is to weigh the bodies, predict their motion in all future time, compute their orbits, determine what changes of form and position these orbits will undergo through thousands of ages, and make maps showing exactly over what cities and towns on the surface of the earth an eclipse of the sun will pass fifty years hence, or over what regions it did pass thousands of years ago. A more hopeless problem than this could not be presented to the ordinary human intellect. There are tens of thousands of men who could be successful in all of the ordinary walks of life, hundreds who could wield empires, thousands who could gain wealth, for one who could take up this astronomical problem with any hope of success. The men who have done it are therefore in intellect the select few of the human race, -- an aristocracy ranking above all others in the scale of being. The astronomical ephemeris is the last practical outcome of their productive genius.

Simon Newcomb (1903) 'Reminiscences of an Astronomer' pp.63-4

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1.0 INTRODUCTION

The satellite system of Saturn has been the subject of much research in celestial mechanics since the foundation of this branch of astronomy in its classical form by Laplace in the late 18th century. It has been described as a "solar system in miniature" by virtue of the range of the types of behaviour which characterise the satellite orbits. Within a single system we have

- A very dominant satellite (Titan) which is unaffected by periodic perturbations of a planetary type by its neighbours, and whose motion features only secular perturbations and small periodic solar perturbations.

- Two pairs of satellites (Mimas - Tethys and Enceladus - Dione) whose mean motions are very nearly in the ratio 2:1, causing (among other things) significant librations in the mean longitudes of the satellites concerned.

- A satellite of rather low mass (Hyperion) whose motion is entirely characterised by a close 3:4 resonance with the dominant satellite in the system. The theory of the motion of Hyperion is a problem of such great complexity that Newcomb placed it second only to the lunar theory.

- A satellite (Iapetus) whose theory is dominated by large periodic solar perturbations due to the great distance at which it orbits Saturn. Moreover, the position of the orbit plane of this satellite is governed by long-period perturbations of roughly equal size acting in two widely-separated planes. This means that the secular theory of the node and inclination of Iapetus is of particular interest.

This diversity of behaviour might at first appear daunting, but each satellite (even Titan) only affects its closest neighbours. Most of the satellites are very small and their perturbing effect is only noticeable when it is amplified by a near-resonance. Thus Tethys perturbs Mimas but not Enceladus or Dione, despite the fact that the latter two are its closest neighbours. We may treat the inner satellites (Mimas, Enceladus, Tethys and Dione) as a self-contained system, and likewise the outer satellites, Rhea, Titan, Hyperion and Iapetus. It is the outer satellite system, and in particular the subset consisting of Titan, Hyperion and Iapetus, that is the subject of this thesis. We choose not to include Rhea as an object for direct study, though we shall always be mindful of its perturbations upon the other three satellites.

We begin in chapter 2 with a revision of Sinclair's (1974) theory of the motion of Iapetus in the light of later critical work by Rapaport (1978) and Sinclair and Taylor (1985). Both of these papers note that Sinclair's theory requires improvement and Rapaport investigates a near-resonance with Titan which affects the mean longitude of Iapetus.

During the course of our revision, we find that the principal omissions from Sinclair's theory arise from solar perturbations in the node and inclination. These perturbations have periods of up to 29 years and affect the observed position of the satellite as seen from the Earth by as much as $0''.14$.

We also find that Rapaport overestimates the significance of the Titan quasi-resonance perturbations by a factor of 3, and we present an improved theory of the motion of Iapetus which includes the additional solar and Titan perturbations. The theory is compared with Sinclair and Taylor's (1985) integration of the motion of the outer satellites to obtain a quantitative estimate of the precision of the theory.

Chapter 3 contains a study of the secular motion of the orbit plane of a satellite acted upon by several perturbing forces in different fixed planes. We find that the concept of the Laplacian plane may easily be extended to any number of perturbing forces up to fourth order in the inclinations. We use auxiliary parameters $p = \sin i \sin \Omega$ and $q = \sin i \cos \Omega$ to represent the position of the orbit plane of the perturbed satellite in an arbitrary fixed reference frame and we show that the pole of the orbit plane describes an ellipse about a point which is the pole of the Laplacian plane of the orbit.

This method is applied to the particular case of Iapetus, which is subject to significant perturbations by Titan and the Sun, and smaller perturbations due to the oblateness of Saturn. The orbit of Iapetus is shown to maintain an almost constant inclination of 7° to its Laplacian plane, upon which it precesses with a period of 3000 years. We fit this model to observed values of the node and inclination of the orbit of Iapetus and we determine the mass of Titan as a result of the fitting process.

Chapters 4 and 5 are concerned with the problem of modelling the motion of Titan, Hyperion and Iapetus by numerical integration. This approach has been used by Sinclair and Taylor with some success. They fitted an integration to photographic (astrometric) observations over the period 1967 to 1982 and determined values for the initial position and velocity components of each of the satellites plus the J_2 form-factor and the mass of Saturn and the mass of Titan.

In this thesis we attempt to fit a similar integration to visual (micrometric) observations made during the period 1874 to 1947. Chapter 4 gives an account of the preparation of the raw data for comparison with any dynamical model. This preparation includes the reduction of the various timescales to Universal Time and Ephemeris Time, the calculation of the topocentric position vector of Saturn at the instant of each observation, and an analysis of the effects of stellar aberration and atmospheric refraction upon position angle and separation measures. In addition, we develop the partial derivatives of position angle and separation observations with respect to the Saturnicentric rectangular coordinates of the satellite(s) involved.

Chapter 5 contains a description of the numerical integration method and the procedure employed in fitting the integration to the observations.

It also contains an account of the results of a number of trial iterations in which we attempt to determine the parameters of the satellite system. In particular, values are obtained for the J_2 form-factor of Saturn and the mass of Titan.

2.0 THE THEORY OF THE MOTION OF IAPETUS

2.1 INTRODUCTION

The motion of Iapetus, the ninth major satellite of Saturn, is characterised by significant Solar perturbations and by the large inclination of the orbit to the equator plane of the primary. The dominant perturbing forces upon Iapetus are, in decreasing order of magnitude :

- The Sun
- Titan
- The oblateness of Saturn

The theory of Iapetus was developed first by H. Struve (1888) whose model includes periodic Solar perturbations plus secular terms in the node, apse, inclination and eccentricity. Sinclair (1974) revised Struve's theory by adding perturbations due to Titan and the oblateness of Saturn, plus extra Solar terms. Sinclair's aim was to include all terms greater than $0^{\circ}.001$ in Saturnicentric position, which corresponds to $0''.01$ seen from Earth at 8.5AU, in order to make effective use of post-1967 photographic observations of the positions of the satellites of Saturn.

Rapaport (1978) suggested that Iapetus is affected by a close commensurability of its mean motion with that of Titan. The mean motions are very nearly in the ratio 5:1 so that $5n_I - n_T$ is $0^\circ.113$ per day. Rapaport added several terms to Sinclair's theory containing the angle $5\lambda_I - \lambda_T$ in the argument. He also added a Solar term to the eccentricity and made a new determination of the mean motion of Iapetus.

The most recent theory of Iapetus is by Harper et al (in submission). We have added several significant Solar terms, notably in the node, and we have evaluated the overall size of the 5:1 Titan perturbations, calculating them in a different manner to Rapaport and finding them to be far smaller than Rapaport suggests.

2.2 USE OF THE NUMERICAL INTEGRATION AS A REFERENCE MODEL

In the current work, Sinclair's (1974) theory of Iapetus was chosen as a basis for further development since it is the most recent full theory and it was constructed using standard techniques involving classical orbital elements. It was thus a relatively easy task to add extra perturbation terms without re-casting the entire theory. Indeed, Sinclair's theory is an extension of that of H. Struve.

Following Sinclair and Taylor (1985), the analytic theory of Iapetus was compared to elements derived from a numerical integration of the motions of Titan, Hyperion and Iapetus over a period of 50 years. The numerical integration has several properties which make it ideal as a reference model of this kind :

1. Osculating elements can be obtained from the integration at regularly spaced dates over a long period. By contrast, observational data are irregularly scattered over the period 1870 to 1983, occurring in small clusters around each opposition. Data are totally absent between 1930 and 1967.

   In addition, it is impossible to extract information about osculating elements directly from observations whereas a numerical integration provides elements with little trouble via instantaneous position and velocity vectors.

2. A numerical integration which has been fitted to observations (Sinclair and Taylor (1985)) is a dynamically consistent representation of the real satellite system. The force model of the integration is not a truncated approximation (as is the case with the disturbing function of an analytical theory) and thus the integration implicitly contains all periodic perturbations limited only by the accuracy of the coordinates produced by the integration calculations.

Comparison of the elements from the analytical theories with those obtained from the integration indicates periodic terms which may have been omitted from the theories. Knowledge of the periods of these terms and their approximate amplitudes enables them to be identified in the expansion of the disturbing function.

The numerical integration used in this work is that of Sinclair and Taylor (1985). Its force model includes Solar perturbations, the second and fourth harmonics of the gravity field of Saturn, and mutual satellite perturbations including those due to Rhea (though Rhea's effect upon Iapetus is negligible and is effectively a small augmentation of Saturn's J_2 coefficient). The numerical integration method is described in detail in chapter 5.

2.3 DISCUSSION OF RESIDUALS FROM SINCLAIR'S THEORY

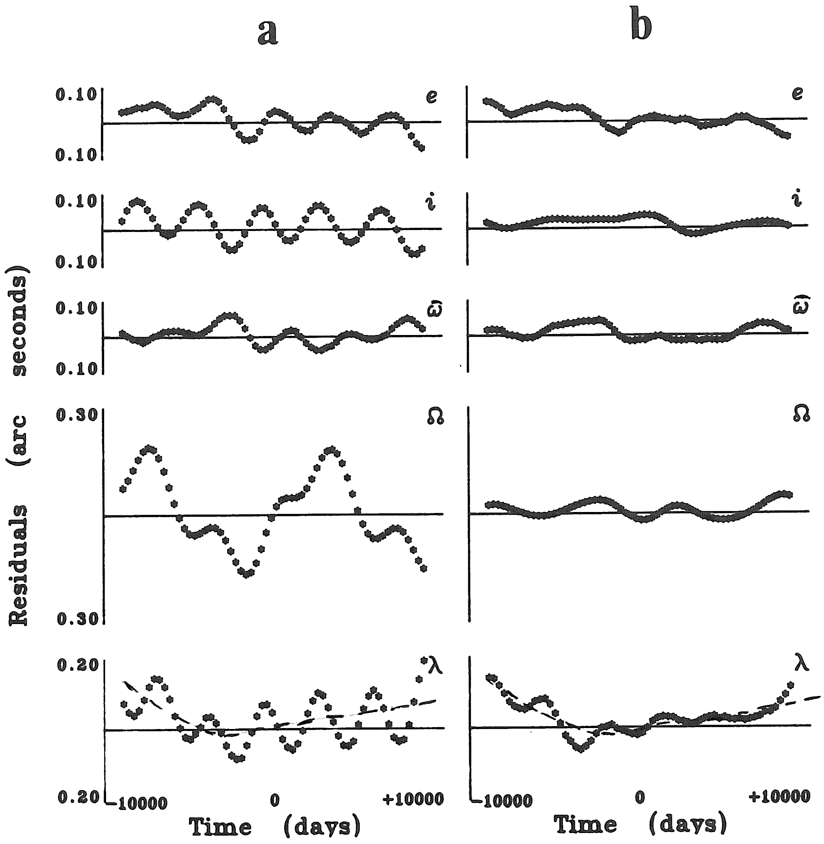


Figure 1. Integration-minus-theory residuals

In Figure 1 we show graphs of the residuals in the eccentricity, apse, inclination, node and mean longitude of Iapetus. The elements are referred to the mean Equator and Equinox of B1950 and the residuals are in the sense Integration-minus-Theory. Thus they represent the terms to be added to the theory in order to make it agree with the integration. The residuals are formed by taking the difference Integration - Theory at intervals of 15 days and calculating average values every 300 days from groups of 20 points. By this method we may eliminate terms of short period (less than 150 days) to enable the long-period behaviour of the elements to be seen more clearly.

The residuals are scaled so that the graphs show the effect upon the observed positions of the satellite at a mean opposition distance of 8.5 AU. In particular, this means that the graphs labelled 'Apse' and 'Node' are plots of $e\Delta\varpi$ and $\sin i\ \Delta\Omega$ respectively and hence they can be compared directly with the graphs of Δe and Δi. Column (a) shows the residuals between the integration and the theory of Sinclair (1974) whilst column (b) shows the residuals between the integration and the improved theory developed in this chapter.

All the elements in column (a) show residuals which are of long period. The period of the residuals in the eccentricity, inclination and apse is 3500 days while that of the node is around 10000 days. Recalling that the mean orbital period of Saturn is 10759 days, we may immediately identify these residuals as Solar perturbations. The terms we seek in the Solar disturbing function have arguments which contain the mean longitude of the Sun but not that of Iapetus. The derivation of these Solar terms is given in the next section.

The most noticeable feature of the perturbations in the mean longitude is a periodic term with a period of approximately 3000 days. It is superimposed upon a term which is secular or of very long period (indicated with a dotted line). The periodic term closely matches the period of the 5:1 Titan perturbations discussed by Rapaport. This is investigated further in a subsequent section, where we shall show that the introduction of 5:1 Titan perturbations reduces the residuals in the mean longitude.

2.4 SOLAR PERTURBATIONS UPON IAPETUS

As a first step in calculating the perturbations upon Iapetus due to the Sun, we must develop an expansion for the disturbing function of the Sun. This is given by the following expression (for the derivation, refer to appendix C).

[1] $$R_s = GM_s\left(\frac{1}{\Delta_s} - \frac{\underline{r}.\underline{r}_s}{r_s^3}\right)$$

where
- G = gravitational constant
- M_s = mass of the Sun
- Δ_s = the distance between the Sun and Iapetus
- $\underline{r}$ = the Saturnicentric position vector of Iapetus
- $\underline{r}_s$ = the Saturnicentric position vector of the Sun
- $r_s = |\underline{r}_s|$

We may write this as

[2] $$R_s = GM_s/r_s \left\{(1 + (r/r_s)^2 - 2\,(r/r_s)\cos X)^{-1/2} - (r/r_s)\cos X\right\}$$

$$= GM_s/r_s \sum_{p=1}^{\infty} (r/r_s)^p\, P_p(\cos X)$$

where $P_p(x)$ is the Legendre polynomial of order p.

Now r/r_s is a small quantity of order 0.0025, so we need only take the first term. We write

[3] $$R_s = GM_s/r_s\, (r/r_s)^2\, (-\tfrac{1}{2} + \tfrac{3}{2}\cos^2 X).$$

By Kepler's third law, we have

[4] $n_s^2 a_s^3 = GM_s$

and we may re-write the expression for R_s as

[5] $R_s = n_s^2 a^2 (r/a)^2 (a_s/r_s)^3 (-\frac{1}{2} + \frac{3}{2}\cos^2 X).$

We must now express R_s in terms of the orbital elements of Iapetus and the Sun. The expansions of powers of the radii vectores are straightforward and may be found in Brouwer and Clemence (1961). In order to evaluate $\cos^2 X$ we consider the orbit planes of Iapetus and the Sun referred to the ecliptic and equinox of B1950.0 as in the accompanying figure.

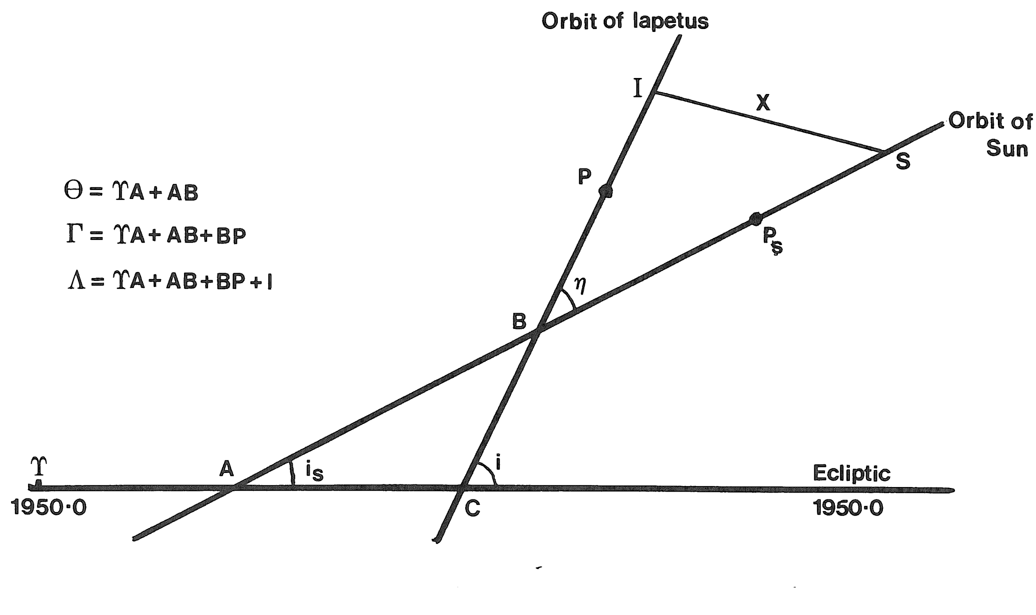


Figure 2. The orbits of Iapetus and the Sun

I and S denote Iapetus and the Sun respectively and P and P_s are the pericentres of Iapetus and the Sun. The notation is as follows.

$g = BP \qquad g_s = BP_s$

$f = PI \qquad f_s = P_s S$

$\vartheta = CB \qquad \Theta = \Omega_s + AB$

$\Gamma = \Omega_s + AB + BP = \Theta + BP$

$\omega = \Omega + CP \qquad \omega_s = \Omega_s + AP_s$

From the formulae of spherical trigonometry we have

[6] $$\cos X = \cos(g + f)\cos(g_s + f_s) + \cos\eta \sin(g + f)\sin(g_s + f_s)$$

where f = the true anomaly of Iapetus

g = the argument of the apse of Iapetus i.e. the arc of the orbit from the ascending node upon the reference plane to the apse

f_s, g_s are the corresponding quantities for the Sun

η = the inclination of the orbit of Iapetus to that of the Sun.

We may rewrite this in terms of the mean anomalies and eccentricities by substituting the equation of the centre for both bodies (see for example Brouwer and Clemence (1961)). Substitution of the elliptic expansions and evaluation of the resulting series for R is a lengthy process and so the computer algebra package CAMAL-F was employed. CAMAL-F is designed to handle the Fourier series of celestial mechanics and it includes facilities for making substitutions of the form

[7] $$\vartheta = \psi + \text{a Fourier series}$$

of which the equation of the centre is an example. The CAMAL-F program for expansion of the Solar disturbing function is given in Appendix A.

We seek those terms in the disturbing function which contain the mean anomaly of the Sun but not that of Iapetus. In addition to the terms given by Sinclair (1974) we find six significant terms.

[8] $$R_1 = (9/8)n_s^2\ a^2\ e_s^2\ (1 - 6\gamma^2 + 6\ \gamma^4)\ \cos 2\ell_s$$

[9] $$R_2 = (105/16)n_s^2\ a^2\ e_s\ e^2\ (1 - 2\gamma^2)\ \cos(3\ell_s + 2\ g_s - 2g)$$

[10] $$R_3 = (51/4)n_s^2\ a^2\ e_s^2\ \gamma^2\ (1 - \gamma^2)\ \cos(4\ell_s + 2g_s)$$

[11] $$R_4 = (21/4)n_s^2\ a^2\ e_s\ \gamma^2\ (1 - \gamma^2)\ \cos(3\ell_s + 2g_s)$$

[12] $$R_5 = -(3/4)n_s^2\ a^2\ e_s\ \gamma^2(1 - \gamma^2)\ \cos(\ell_s + 2g_s)$$

[13] $$R_6 = (3/4)n_s^2\ a^2\ e_s\ (1 - 6\ \gamma^2 + 6\ \gamma^4)\ \cos\ \ell_s$$

where we have written $\gamma = \sin(\eta/2)$

In order to simplify the calculation of the perturbations, we use elements of the orbit of Iapetus referred to the orbit plane of the Sun about Saturn. We define elements Λ, Γ and Θ corresponding to mean longitude, apse and node as in Figure 2 on page 13. The Lagrange planetary equations become

[14] $$\frac{de}{dt} = -\frac{1}{na^2}\frac{\sqrt{1-e^2}}{e}\frac{\partial R}{\partial \Gamma}$$

[15] $$\frac{d\Gamma}{dt} = \frac{1}{na^2}\left\{\frac{\sqrt{1-e^2}}{e}\frac{\partial R}{\partial e} + \frac{\gamma}{2}\frac{\partial R}{\partial \gamma}\right\}$$

[16] $$\frac{d\Lambda}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\tan \eta/2}{na^2}\frac{\partial R}{\partial \eta}$$

[17] $$\frac{d\eta}{dt} = -\frac{1}{na^2}\frac{1}{2\gamma\sqrt{1-\gamma^2}}\frac{\partial R}{\partial \Theta}$$

[18] $$\frac{d\Theta}{dt} = \frac{1}{na^2}\frac{1}{4\gamma}\frac{\partial R}{\partial \gamma}$$

where only the lowest power of the eccentricity has been retained. We express the arguments of the disturbing function as functions of Λ, Γ and Θ using

[19] $$\ell = \Lambda - \Gamma \qquad \ell_s = \lambda_s - \varpi_s$$
$$g = \Gamma - \Theta \qquad g_s = \varpi_s - \Theta$$

and we calculate the derivatives of the elements substituting each term of the disturbing function into the planetary equations. We assume all the elements on the right hand side of the equations to be constants, except for the mean longitudes which are assumed to vary at constant rates. Upon integration, we obtain the perturbations in e, Γ, η and Θ which we denote by Δe, $\Delta\Gamma$, $\Delta\eta$, $\Delta\Omega$

The form of the perturbation in the eccentricity is independent of the choice of reference plane and so Δe may be quoted directly. However, in order to obtain expressions for the perturbations in the node, apse and inclination with respect to the ecliptic and equinox of B1950 we must apply transformations to $\Delta\Gamma$, $\Delta\eta$ and $\Delta\Omega$. Consider the spherical triangle ABC in Figure 2 on page 13. We may write

$$[20] \quad \begin{aligned} \cos i &= \cos i_s \cos \eta - \sin i_s \sin \eta \cos (\Theta-\Omega_s) \\ \sin i \cos (\Omega-\Omega_s) &= \sin i_s \cos \eta + \cos i_s \sin \eta \cos (\Theta-\Omega_s) \\ \sin i \sin (\Omega-\Omega_s) &= \sin \eta \sin (\Theta-\Omega_s) \end{aligned}$$

from which may be obtained the following derivatives (see appendix B) :

$$[21] \quad \begin{aligned} \partial i/\partial\eta &= + \cos \vartheta \quad ; & \partial i/\partial\Theta &= - \sin \vartheta \sin \eta \\ \sin i\ \partial\Omega/\partial\eta &= + \sin \vartheta \quad ; & \sin i\ \partial\Omega/\partial\Theta &= + \cos \vartheta \sin \eta \\ \sin i\ \partial\vartheta/\partial\eta &= - \sin \vartheta \cos i \quad ; & \sin i\ \partial\vartheta/\partial\Theta &= + \sin i_s \cos (\Omega-\Omega_s) \end{aligned}$$

and hence to first order we may write

$$[22] \quad \begin{aligned} \Delta i &= \cos \vartheta\ \Delta\eta - \sin \vartheta \sin \eta\ \Delta\Theta \\ \sin i\ \Delta\Omega &= \sin \vartheta\ \Delta\eta + \cos \vartheta \sin \eta\ \Delta\Theta \\ \sin i\ \Delta\vartheta &= - \sin \vartheta \cos i\ \Delta\eta + \sin i_s \cos (\Omega-\Omega_s)\ \Delta\Theta. \end{aligned}$$

We note also that

$$[23] \quad \varpi = \Gamma - \Theta + \Omega + \vartheta$$

therefore

$$[24] \quad \Delta\varpi = \Delta\Gamma - \Delta\Theta + \Delta\Omega + \Delta\vartheta.$$

From Figure 1 on page 10, we expect the dominant perturbation in the eccentricity and apse to have a period which is one third that of the Sun. Consequently, its argument must contain the angle $3\ell_s$. We notice also from the Lagrange planetary equation for de/dt that in order for a term to contribute to Δe, its argument must contain Γ, and hence in the original expression for R_s, it must contain g, the apse of Iapetus. The only term satisfying these requirements is R_2. From this term we find

$$[25] \qquad \Delta e = (35n_s/8n)\, e\, e_s \sqrt{1 - e^2}\, (1 - 2\gamma^2) \cos(3\ell_s + 2g_s - 2g)$$

$$[26] \qquad e\, \Delta\varpi = (35n_s/8n)\, e\, e_s \sqrt{1 - e^2}\, (1 - 2\gamma^2) \sin(3\ell_s + 2g_s - 2g).$$

We now calculate the perturbations in the node and inclination. From equation [22] we see that Δi and $\sin i\, \Delta\Omega$ can be expressed as a combination of $\Delta\eta$ and $\Delta\Theta$. Consider a term in the disturbing function with argument Ψ. Its contribution to $\Delta\eta$ may be written

$$[27] \qquad \Delta\eta = A \cos \Psi$$

and $\Delta\Theta$ may be written

$$[28] \qquad \Delta\Theta = B \sin \Psi$$

where A and B are functions of n, n_s, e, e_s and γ obtained from the Lagrange planetary equations. They may be treated as constants. Substituting into equation [22] we obtain

$$[29] \qquad \begin{aligned} \Delta i &= \tfrac{1}{2}(A + B \sin \eta) \cos (\Psi + \vartheta) + \tfrac{1}{2}(A - B \sin \eta) \cos (\Psi - \vartheta) \\ \sin i\, \Delta\Omega &= \tfrac{1}{2}(A + B \sin \eta) \cos (\Psi + \vartheta) + \tfrac{1}{2}(A - B \sin \eta) \cos (\Psi - \vartheta). \end{aligned}$$

As an example we consider the term R_4

$$R_4 = 21/4 \quad n_s^2 a^2 e_s \gamma^2 (1 - \gamma^2) \cos (3\ell_s + 2g_s).$$

We substitute equations [19] into the argument of the term :

[30] $$\Psi = 3\ell_s + 2g_s = 3\lambda_s - \omega_s - 2\Theta$$

then

[31] $$d\Theta/dt = 21/8 \quad (n_s^2/n)\, e_s\, (1 - 2\gamma^2) \cos (3\ell_s + 2g_s)$$

giving

[32] $$\Delta\Theta = 7/8 \quad (n_s/n)\, e_s\, (1 - 2\gamma^2) \sin (3\ell_s + 2g_s)$$

and

[33] $$d\eta/dt = -21/4 \quad (n_s^2/n)\, e_s\, (1 - \tfrac{1}{2}\gamma^2) \sin (3\ell_s + 2g_s)$$

giving

[34] $$\Delta\eta = 7/8 \quad (n_s/n)\, e_s\, (1 - \tfrac{1}{2}\gamma^2) \cos (3\ell_s + 2g_s)$$

so we may write

[35]
$$A = 7/4 \quad (n_s/n)\, e_s \gamma (1 - \tfrac{1}{2}\gamma^2)$$
$$B = 7/8 \quad (n_s/n)\, e_s \gamma (1 - 2\gamma^2).$$

The terms R_1, R_3, R_5 and R_6 are treated in exactly the same way. We present the coefficients A and B for each of the terms in tabular form below.

Term	Argument (Ψ)	A	B
R_1	$2\ell_s$	0	$-(27n_s/16n)\ e^2_s$
R_3	$4\ell_s + 2g_s$	$(51n_s/16n)\ \gamma\ e^2_s$	$(51n_s/32n)\ e^2_s$
R_4	$3\ell_s + 2g_s$	$(7n_s/4n)\ e_s\ \gamma(1 - \frac{1}{2}\gamma^2)$	$(7n_s/8n)\ e_s\ (1 - 2\gamma^2)$
R_5	$\ell_s + 2g_s$	$-(3n_s/4n)\ e_s\ \gamma(1 - \frac{1}{2}\gamma^2)$	$-(3n_s/8n)\ e_s\ (1 - 2\gamma^2)$
R_6	ℓ_s	0	$-(9n_s/4n)\ e_s$

We adopt the following values for the elements of the orbits of the Sun and Iapetus. The elements are referred to the mean Ecliptic and Equinox of B1950.0. and are for the epoch JD 2409786.0 (1885.67). These elements change very slowly and the coefficients of the terms are rather small so we do not introduce significant errors by adopting fixed values for the nodes, inclinations and eccentricities. For the Sun

$n_s = 0°.0334597$ /day $\qquad e_s = 0.05560$

$i_s = 2°.4909 \qquad \Omega_s = 113°.158$

and for Iapetus

$n = 4°.53795711$ /day $\qquad e_o = 0.028796$

$i_o = 18°.4606 \qquad \Omega_o = 143°.1209$

From the spherical triangle ABC in Figure 2 on page 13 we may write

$$\begin{aligned} \sin \eta \sin \vartheta &= \sin i_s \sin (\Omega_o - \Omega_s) \\ \sin \eta \cos \vartheta &= \cos i_s \sin i_o - \sin i_s \cos i_o \cos (\Omega_o - \Omega_s) \qquad [36] \\ \cos \eta &= \cos i_s \cos i_o + \sin i_s \sin i_o \cos (\Omega_o - \Omega_s) \end{aligned}$$

and we obtain $\eta = 16°.348$, $\gamma = 0.14218$, $\vartheta = 4°.423$.

Since ϑ is a small angle, we may neglect it when calculating the coefficients of the perturbations in i and Ω. The coefficients themselves are rather small and the effect of assuming $\vartheta = 0$ will be negligible. It has the advantage of reducing the number of terms to be added to the expressions for the perturbations in i and Ω, since instead of two terms with arguments $\Psi + \vartheta$ and $\Psi - \vartheta$ we will have only one term with argument Ψ.

We present below the terms to be added to Sinclair's (1974) theory as a result of this work.

[37] $\Delta e = +0.00000496 \cos (3\ell_s + 2g_s - 2g)$

[38] $e\,\Delta\omega = +0.00000496 \sin (3\ell_s + 2g_s - 2g)$ radians

[39] $\Delta i = +0^\circ.0005 \cos (4\ell_s + 2g_s) + 0^\circ.0058 \cos (3\ell_s + 2g_s)$

$-0^\circ.0024 \cos (\ell_s + 2g_s)$

[40] $\sin i\,\Delta\Omega = -0^\circ.0006 \sin 2\ell_s + 0^\circ.0003 \sin (4\ell_s + 2g_s)$

$+ 0^\circ.0028 \sin (3\ell_s + 2g_s) - 0^\circ.0012 \sin (\ell_s + 2g_s) - 0^\circ.0142 \sin \ell_s$

2.5 THE 5:1 QUASI-RESONANCE DUE TO TITAN

Comparison of Sinclair's (1974) theory of Iapetus with the numerical integration reveals a periodic residual in the mean longitude with an amplitude of 0".1 arc-seconds and a period of 3000 days. This may be identified with a 5:1 quasi-resonance due to Titan which was first noted by Plana (1826). It arises from the fact that the mean motion of Titan is very nearly five times that of Iapetus.

[41] $5n_I - n_T = 0^\circ.113$ per day

Thus

[42] $\nu = n_I/(5n_I - n_T) = 40.2$

which suggests that terms which include $5\lambda_I - \lambda_T$ in their argument may give rise to significant perturbations in the mean longitude since the coefficients of such terms include the square of this factor.

This was developed by Rapaport (1978) who investigated a number of these terms using a method attributed to J.L. Simon. Rapaport concluded that such terms may have Saturnicentric amplitudes up to 90 arc-seconds which corresponds to 0".25 as seen from the Earth at 8.5 AU. The residual plot of the mean longitude in Figure 1 on page 10 (column (a)) shows an amplitude of only 0".1 as seen from 8.5 AU and this leads us to suspect that Rapaport's work overestimates the net size of the quasi-resonance terms and gives a somewhat misleading impression of their importance.

We set out to re-calculate these terms using the method employed by Sinclair in his (1974) revision of the theory of Iapetus.

2.5.1 THE DISTURBING FUNCTION

We seek all terms in the disturbing function of Titan upon Iapetus which include $5\ell - \ell_T$ in the argument. (ℓ denotes the mean anomaly of Iapetus and ℓ_T the mean anomaly of Titan). We use the expansion of the planetary disturbing function developed by Newcomb in volume 5 of the "Astronomical Papers for the Use of the American Ephemeris". Using Newcomb's notation, we find that the following sets of indices give suitable terms :

(1)	k = 2	j = 0	j' = 0	i = 3
(2)	k = 1	j = 2	j' = 0	i = 4
(3)	k = 1	j = 1	j' = 1	i = 3
(4)	k = 1	j = 0	j' = 2	i = 2

whence

[43] $$R_1=(3/8)n^2a^2m_T\ \gamma^4(\alpha^2 b_{5/2}^{(3)} - \frac{5}{2}\ \gamma^2\alpha^3(b_{7/2}^{(2)} + b_{7/2}^{(4)}))\ \cos(5\ell-\ell_T+5g_1-g_T)$$

[44] $$R_2=(1/16)n^2a^2m_T\ e\ _T^{\ 2}\ \gamma^2\ \alpha(31 + 12\alpha D_\alpha + \alpha^2 D_\alpha^2\ )\ b_{5/2}^{(4)}\ \cos(5\ell-\ell_T+5g_1-3g_T)$$

[45] $$R_3= -((1/8)n^2a^2m_T\ e\ e_T\ \gamma^2\ \alpha(50 + 16\alpha D_\alpha + \alpha^2 D_\alpha^2\ )\ b_{5/2}^{(3)}\ \cos(5\ell-\ell_T+4g_1-2g_T)$$

[46] $$R_4=(1/16)n^2a^2m_T\ e^2\ \gamma^2\ \alpha(85 + 20\alpha D_\alpha + \alpha^2 D_\alpha^2\ )\ b_{5/2}^{(2)}\ \cos(5\ell-\ell_T+3g_1-g_T)$$

where γ = sine of half the mutual inclination of the orbits of Titan and Iapetus

α = ratio of the semi-major axes of Titan to Iapetus = 0.34303

D_α = the differential operator $d/d\alpha$

$b_s^{(i)}$ are Laplace coefficients.

These terms have been verified by deriving them independently from Pierce's (1849) expansion of the planetary disturbing function.

2.5.2 CALCULATION OF THE PERTURBATION IN THE MEAN LONGITUDE

We follow the method of Sinclair (1974) and calculate the perturbations of the elements of the orbit of Iapetus referred to the orbit plane of Titan. This introduces an error since the orbit plane of Titan is not fixed but varies slowly due to the influence of the Sun and the oblateness of Saturn. However, the inertial terms may be neglected in this work since they are very small.

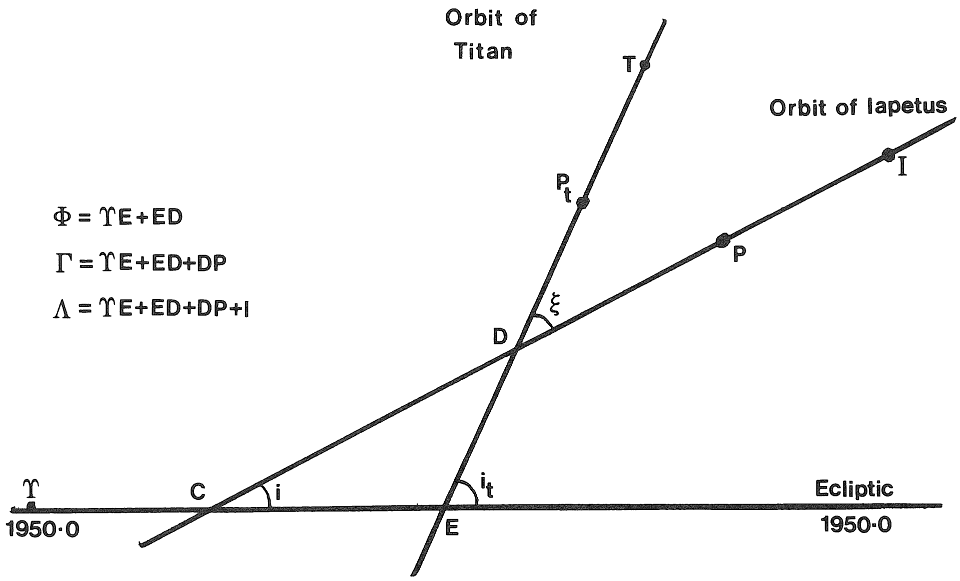


Figure 3. Reference frame for Titan perturbations

The notation is as follows :

$\Omega_T = ♈E$	$\omega_T = ♈E + EP_T$
$\phi = CD$	$\Phi = ♈E + ED$
$g_1 = DP$	$g_T = DP_T$

The relevant Lagrange planetary equations are

[47] $$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \Lambda}$$

[48] $$\frac{d\Lambda}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\tan \xi/2}{na^2}\frac{\partial R}{\partial \xi}$$

[49] $$\frac{d\Theta}{dt} = \frac{1}{na^2 \sin\xi} \frac{\partial R}{\partial \xi}$$

[50] $$\frac{d\xi}{dt} = \frac{\tan \xi/2}{na^2} \frac{\partial R}{\partial \Lambda} - \frac{1}{na^2 \sin\xi} \frac{\partial R}{\partial \Theta}$$

and the perturbation in the mean longitude referred to the Ecliptic and Equinox of B1950 may be written

[51] $$\Delta\lambda = \Delta\Lambda - \Delta\Theta + \Delta\Omega + \Delta\phi$$

where (cf. Solar perturbations)

[52] $$\sin i\ (\Delta\Omega + \Delta\phi) = \sin\phi\ (1 - \cos i)\ \Delta\xi + (\sin\xi \cos\phi + \sin i_T \cos(\Omega - \Omega_T))\Delta\Theta.$$

We note that the mean motion n in the equation for $d\Lambda/dt$ must include the perturbations from Δa : since n^2a^3 is a constant we have

[53] $$\Delta n/n = -3/2 \quad \Delta a/a$$

or

[54] $$dn/dt = -3/a^2 \quad \partial R/\partial\Lambda.$$

Thus the first part of $\Delta\Lambda$ contains the double integral

[55] $$\iint \frac{dn}{dt}\, dt^2 = -\frac{3}{a^2}\iint \frac{\partial R}{\partial \lambda}\, dt^2$$

If we denote the mean rate of change of the argument of a given term in R by κ then the process of integrating $\partial R/\partial\Lambda$ twice with respect to time will introduce a factor $1/\kappa^2$. In the case of the 5:1 terms under consideration, κ is a small quantity and hence $1/\kappa^2$ will be large. We expect the double integration of dn/dt to yield the most significant part of $\Delta\Lambda$ for such terms.

In the next section we derive the perturbation in λ from a general 5:1 term in the disturbing function.

2.5.3 DERIVATION OF $\Delta\lambda$ FROM ANY 5:1 TERM

Consider any of the 5:1 terms in R given in section 2.5.1; we may write it as

[56] $$R = n^2 a^2 \mu_T \beta \sin (5\ell - \ell_T + jg - j_T g_T)$$

where β is a dimensionless function of e, e_T, γ and α

j , j_T are integers.

We may split $\Delta\lambda$ (equation [51]) into four parts :

1. A term arising from the double integration of dn/dt

2. A term arising from the double integration of $d\Lambda/dt - n$

3. The term $-\Delta\Theta$ in equation [51] plus the contribution of $\Delta\Theta$ to $\Delta\Omega+\Delta\phi$ (equation [52])

4. The contribution of $\Delta\xi$ to $\Delta\Omega+\Delta\phi$.

We may write accordingly

[57] $$\Delta\lambda = \Delta(\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} + \lambda^{(4)}).$$

It is instructive and convenient to treat each part separately. We first make the substitutions

$$\ell = \Lambda - \Gamma \qquad g = \Gamma - \Theta$$

$$\ell_T = \lambda_T - \tilde{\omega}_T \qquad g_T = \tilde{\omega}_T - \Theta$$

into the argument of the term, which becomes

$$5\Lambda + (j - 5)\Gamma + (j_T - j)\Theta - \lambda_T + (1 - j_T)\tilde{\omega}_T.$$

Inspection of the Lagrange planetary equations shows that we shall only be concerned with the coefficients of Λ and Θ since we are not required to evaluate the derivatives of the argument with respect to any of the other angles. We note, therefore, that the coefficient of Λ is 5 and that of Θ is $j_T - j$, and we may resume writing the argument in terms of ℓ, ℓ_T, g and g_T. For brevity, we denote the argument of each term simply as $\Psi = 5\ell - \ell_T + jg - j_T g_T$ and we denote the important ratio $n/(5n - n_T)$ by ν. The value of ν is approximately 40.

Term (1)

The mean motion is dependent upon the semi-major axis by virtue of Kepler's third law, which implies that $n^2 a^3$ is constant for each satellite. Thus a perturbation in the semi-major axis requires a balancing perturbation in the mean motion. Thus we have

[58] $$2\ \Delta n/n = -3\ \Delta a/a$$

from which we may obtain

[59] $$d^2\lambda^{(1)}/dt^2 = dn/dt = -3/a^2\ \partial R/\partial\Lambda$$

which upon integration twice yields

[60] $$\Delta\lambda^{(1)} = -15\ \nu^2 \mu_T \beta \sin \Psi$$

where, as we have said, Ψ is the argument of the term. This part of $\Delta\lambda$ will later be shown to be the largest since it contains the factor ν^2 which augments the other factors in the coefficient by three orders of magnitude.

Term (2)

The remainder of $d\Lambda/dt$ after removing n (which we have dealt with as term (1)) gives us $d\lambda^{(2)}/dt$:

$$[61] \quad \frac{d\lambda^{(2)}}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{e}{2na^2}\frac{\partial R}{\partial e} + \frac{\gamma}{2na^2}\frac{\partial R}{\partial \gamma}$$

where $\gamma = \sin \xi/2$ replaces ξ as the inclination parameter. In order to evaluate $\partial R/\partial a$ it is more convenient to write R as

$$[62] \qquad R = \frac{k^2\mu_T \, \beta \sin \Psi}{a}$$

using Kepler's third law to write $k^2(M + m_T) = n^2a^3$ but neglecting m_T since it is very small in relation to M, the mass of Saturn.

Then R depends upon the semi-major axis directly, and indirectly via β which is a function of $\alpha = a_T/a$. Clearly,

$$[63] \quad \frac{\partial R}{\partial a} = k^2\mu_T \frac{\partial}{\partial a}\left(\frac{\beta}{a}\right) \sin \Psi$$

and

$$[64] \quad \frac{\partial}{\partial a}\left(\frac{\beta}{a}\right) = -\frac{\beta}{a^2} + \frac{1}{a}\frac{\partial \beta}{\partial a} = -\frac{\beta}{a^2} - \frac{\alpha}{a^2}\frac{\partial \beta}{\partial \alpha}$$

$$= -\frac{1}{a^2}\left(\beta + \alpha\frac{\partial \beta}{\partial \alpha}\right)$$

The derivatives of R with respect to e and γ are straightforward since the dependence is solely via β. Hence we have

$$[65] \quad \frac{d\lambda^{(2)}}{dt} = n\mu_T \left\{ \alpha\left(\beta + \alpha\frac{\partial\beta}{\partial\alpha}\right) + \frac{e}{\alpha}\frac{\partial\beta}{\partial e} + \frac{\gamma}{\alpha}\frac{\partial\beta}{\partial\gamma} \right\} \cos \Psi$$

and thus

$$[66] \quad \Delta\lambda^{(2)} = \nu\mu_T \left\{ \alpha\left(\beta + \alpha\frac{\partial\beta}{\partial\alpha}\right) + \frac{e}{\alpha}\frac{\partial\beta}{\partial e} + \frac{\gamma}{\alpha}\frac{\partial\beta}{\partial\gamma} \right\} \sin \Psi$$

Term (3)

We may combine the $-\Delta\Theta$ in equation [51] with the part of $\Delta\Omega+\Delta\phi$ which contains $\Delta\Theta$. Denote this by $\lambda^{(3)}$. Then

[67]

$$d\lambda^{(3)} = n\mu_T \left(-1 + (\sin\xi \cos\phi + \sin i_T \cos(\Omega - \Omega_T))/\sin i\right)\frac{1}{4\gamma}\frac{\partial\beta}{\partial\gamma} \cos \Psi$$

and thus

[68]

$$\Delta\lambda^{(3)} = \nu\mu_T \left(-1 + (\sin\xi \cos\phi + \sin i_T \cos(\Omega - \Omega_T))/\sin i\right)\frac{1}{4\gamma}\frac{\partial\beta}{\partial\gamma} \sin \Psi.$$

Term (4)

The remainder of $\Delta\Omega+\Delta\phi$ has a factor $\Delta\xi$. Denoting this by $\Delta\lambda^{(4)}$ we have

[69] $\Delta\lambda^{\langle 4 \rangle} = \sin\phi\ (1 - \cos i)/\sin i \quad \Delta\xi.$

Thus

[70] $$\frac{d\lambda^{(4)}}{dt} = -n\mu_T \frac{\sin\phi\,(1-\cos i)}{\sin i}\,\beta \sin\Psi$$

and hence

[71] $$\Delta\lambda^{(4)} = \nu\mu_T \frac{\sin\phi\,(1-\cos i)}{\sin i}\,\beta\cos\Psi$$

2.5.4 NUMERICAL VALUES OF THE COEFFICIENTS

We shall use the following numerical values of the constants to evaluate the coefficients of the 5:1 terms.

$\Omega_t - \Omega = 26^\circ.370$	$i_t = 27^\circ.659$
$\alpha = 0.34303$	$e_t = 0.0288$
$\mu_t = 2.412\ 10^{-4}$	$\psi = 36^\circ.1322$

Hence

$\xi = 13^\circ.6964$	$\phi = 60^\circ.5514$

The values of the nodes and inclinations vary over long periods of time, but it is sufficient to adopt fixed values since the variations in the coefficients will be very small.

The Laplace coefficients are evaluated from their power series expansions. We present the contributions from each of the four parts of $\Delta\lambda$ in the table below.

Term	R_1	R_2	R_3	R_4
Argument $-(5\ell-\ell_T)$	$5g - g_T$	$5g - 3g_T$	$4g - 2g_T$	$3g - g_T$
	Coefficient of sine (argument)			
$\Delta\lambda^{\langle 1\rangle}$	-10.13	- 2.96	+17.29	-25.96
$\Delta\lambda^{\langle 2\rangle}$	+ 0.46	+ 0.04	- 2.13	+ 0.49
$\Delta\lambda^{\langle 3\rangle}$	+ 0.75	+ 0.12	- 0.67	+ 1.01
SUM	- 8.92	- 2.80	-14.49	-24.46
	Coefficient of cosine (argument)			
$\Delta\lambda^{\langle 4\rangle}$	- 0.04	- 0.01	+ 0.03	- 0.05

As expected, the most significant part of the coefficient of sine (argument) comes from $\Delta\lambda^{\langle 1\rangle}$. The coefficients of cosine (argument) from $\Delta\lambda^{\langle 4\rangle}$ are entirely negligible and may be ignored. Indeed, if the derivation of these terms had been carried out rigorously i.e. with respect to a fixed reference frame rather than the orbit plane of Titan, then there would be no cosine terms at all in the perturbation in the mean longitude.

We may now write in full the perturbation in the mean longitude due to the 5:1 quasi-resonance with Titan.

[72]
$$\begin{aligned}\Delta\lambda = & -0^\circ.00248 \sin (5\ell - \ell_T + 5g_1 - g_T) \\ & -0^\circ.00078 \sin (5\ell - \ell_T + 5g_1 - 3g_T) \\ & +0^\circ.00403 \sin (5\ell - \ell_T + 4g_1 - 2g_T) \\ & -0^\circ.00679 \sin (5\ell - \ell_T + 3g_1 - g_T)\end{aligned}$$

2.5.5 COMPARISON WITH RAPAPORT'S COEFFICIENTS

Rapaport (1978) carried out his derivation of the 5:1 terms with respect to the equator and equinox of B1950. In his notation we have

ℓ	mean anomaly of Titan
ℓ'	mean anomaly of Iapetus
g ,h	argument of the apse and longitude of the node of Titan
g' ,h'	argument of the apse and longitude of the node of Iapetus

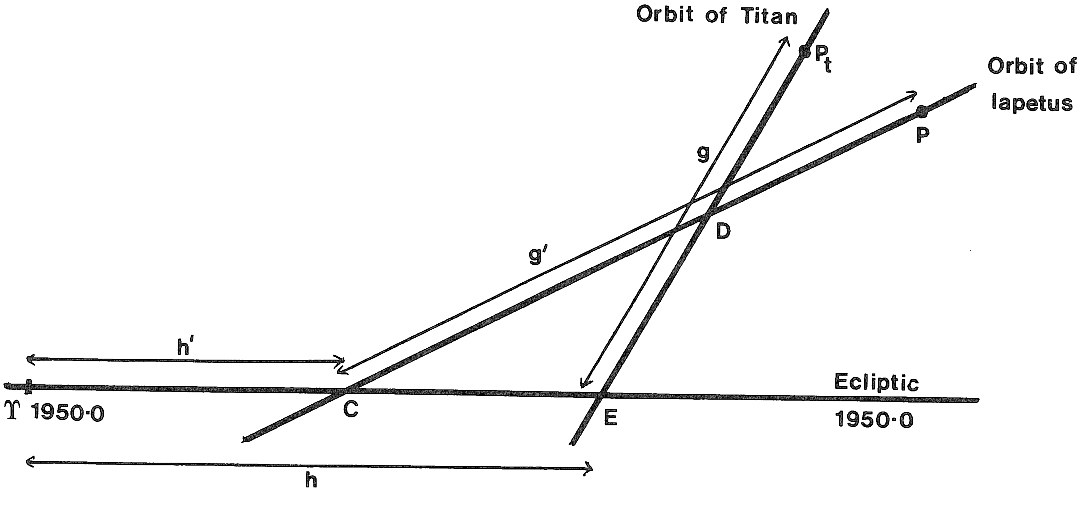


Figure 4. The orbits of Titan and Iapetus

In Rapaport's notation

$$g = EP_T \qquad g' = CP$$

$$h = \Upsilon E \qquad h' = \Upsilon C.$$

In Sinclair's notation

$$g_T = DP_T \qquad g_1 = DP$$

$$\phi = CD \qquad \psi = ED.$$

We may relate Rapaport's notation to that of Sinclair by writing

Rapaport	→	Sinclair
ℓ	→	ℓ_T
ℓ'	→	ℓ
g	→	$\phi + g_T$
g'	→	$\psi + g_1$
h	→	Ω_T
h'	→	Ω

In this context, the symbol → is to be read as 'is replaced by'. Using these substitutions we may transform the arguments of Rapaport's 5:1 terms into a form equivalent to our development. For example, in Rapaport (1978) Table 2 we find the term

$$-42''.64 \sin (5\ell - 5\ell' + g - 5g' + h - h').$$

The argument of this term becomes, upon substitution,

$$\begin{aligned} & \ell_T - 5\ell + \psi + g_T - 5(\phi+g_1) + \Omega_T - \Omega \\ = \quad & \ell_T - 5\ell + g_T - 5g_1 + (\psi - 5\phi + \Omega_T - \Omega). \end{aligned}$$

The first part of this argument varies quite rapidly and may be recognised as the argument of the term in equation [43] and of the first term of equation [72], though with the opposite sign due to Rapaport's notation. The second part may be regarded as a slowly-varying quantity since it contains only angles depending upon the nodes and inclinations of the orbits, which vary over periods of the order of several hundred to several thousand years.

Carrying out this transformation upon the terms given by Rapaport in his Table 2, we notice that the first five terms share the same rapidly-varying part and they correspond to the term in equation [43]. Likewise, Rapaport's terms 6 and 7 have a rapidly-varying part

$$\ell_T - 5\ell + g_T - 3g_1$$

and terms 8, 9 and 10 have a rapidly-varying part

$$\ell_T - 5\ell + 2g_T - 4g_1.$$

Thus they correspond to the terms given in equations [46] and [45].

Consider the first five terms of Rapaport (1978) Table 2. We may write them as

[73] $$\sum_{i=1}^{5} c_i \sin (\ell_T - 5\ell + g_T - 5g_1 + \Psi_i)$$

where c_i is the amplitude given by Rapaport and Ψ_i is the slowly-varying part of the argument, a linear combination of ϕ, ψ and $\Omega_T - \Omega$. Now write

[74] $$C_i = c_i \sin \Psi_i \quad , \quad S_i = c_i \cos \Psi_i .$$

Then [73] may be written as

$$[75] \quad \sum_{i=1}^{5}\left(S_i \sin(\ell_T - 5\ell + g_T - 5g_1) + C_i \cos(\ell_T - 5\ell + g_T - 5g_1)\right)$$

$$= \left(\sum_{i=1}^{5} S_i\right) \sin(\ell_T - 5\ell + g_T - 5g_1) + \left(\sum_{i=1}^{5} C_i\right) \cos(\ell_T - 5\ell + g_T - 5g_1).$$

We may evaluate the coefficients C_i and S_i using values of ϕ, ψ and $\Omega_T - \Omega$ derived from Sinclair (1977) :

$$\Omega = 142^\circ.574 \qquad \iota = 18^\circ.3206$$

$$\Omega_T = 168^\circ.944 \qquad \iota_T = 27^\circ.659$$

and hence

$$\phi = 60^\circ.551 \qquad \psi = 36^\circ.132$$

where all values are for the epoch JD 2415020 and are referred to the ecliptic and equinox of 1950.

We find that Rapaport's terms reduce to

$$+ 12''.28 \sin(\ell_T - 5\ell + g_T - 5g_1) + 3''.98 \cos(\ell_T - 5\ell + g_T - 5g_1).$$

Repeating this operation upon Rapaport's other 5:1 terms yields

$$+ 20''.90 \sin(\ell_T - 5\ell + g_T - 3g_1) - 7''.33 \cos(\ell_T - 5\ell + g_T - 3g_1)$$

and

$$- 15''.99 \sin(\ell_T - 5\ell + 2g_T - 4g_1) - 9''.44 \cos(\ell_T - 5\ell + 2g_T - 4g_1).$$

The following table gives a comparison of Rapaport's 5:1 terms with those derived in this work.

Argument $-(5\ell-\ell_T)$	Our coefft. of sin(arg)	Rapaport's coeffts. sin(arg)	cos(arg)	No. of terms in Rapaport
$5g - g_T$	$-8''.92$	$-12''.28$	$+3''.98$	5
$5g - 3g_T$	$-2''.80$			0
$4g - 2g_T$	$+14''.49$	$+15''.99$	$-9''.44$	3
$3g - g_T$	$-24''.46$	$-20''.90$	$-7''.33$	2

There is good agreement in the magnitudes of the coefficients when they are written in this form. We note that when Rapaport's terms are combined and written in a physically more meaningful way, their amplitudes are far smaller than Rapaport's published coefficients (1978, table 2) would suggest.

2.6 COMPARISON OF THE THEORIES WITH THE NUMERICAL INTEGRATION.

We now compare the theories of Iapetus developed in this chapter with the numerical integration of Sinclair and Taylor (1985). We may identify four variants of the theory of Iapetus :

1. Sinclair (1974).

2. Harper et al (1) : Sinclair (1974) plus solar terms in eccentricity, apse, node and inclination developed in this chapter.

3. Harper et al (2) : Harper et al (1) plus the 5:1 Titan terms in the mean longitude developed in this chapter.

4. Harper-Rapaport : Harper et al (1) plus the 5:1 Titan terms developed by Rapaport (1978).

Each of these theories is fitted to the numerical integration by a process of repeated corrections to the elements of the orbit of Iapetus. Thus the comparison shows how closely the chosen theory represents the reference integration, rather than how closely it represents the real orbit of Iapetus. However, the integration itself has been obtained by fitting to photographic observations over a period of some 15 years. Moreover, as explained in the introduction to this chapter, the integration may be regarded as a dynamically consistent representation of a satellite system which closely resembles the real system over a limited span of time. We may expect that a comparison of the various theories with this integration will provide an indication of the precision of such theories when we eventually test them against observational data.

The process of fitting the theories to the integration yields the residual differences between the position of Iapetus given by the integration and the position given by the chosen theory. If we write

$\underline{r}_i$ = position of Iapetus at any time given by the numerical integration

$\underline{r}_t$ = position of Iapetus at the same time given by the theory

then the residual of greatest interest in the comparison is

$$s = | \underline{r}_i - \underline{r}_t |.$$

As part of the fitting process, we obtain values of s at 1216 regularly-spaced dates across the 50-year span of the integration. We form the root-mean-square of these values and we note the maximum value for each theory once the fitting process has converged. In the table below we give the values of the RMS and maximum residual for each of the theories.

Theory	Root Mean Square Residual (A.U.)	Maximum Residual (A.U.)
Sinclair	0.00000543	0.00002023
Harper et al (1)	0.00000401	0.00001586
Harper et al (2)	0.00000348	0.00001311
Harper/Rapaport	0.00000382	0.00001387

Comparison of the values for Sinclair and for Harper et al (1) shows the significant improvement made by the addition of the Solar perturbations. Both the RMS and maximum residuals are reduced by a quarter. This is principally due to the term in the node

$$\sin \iota \, \Delta\Omega = -0^\circ.0142 \sin \ell_s .$$

The maximum effect of such a term upon the Saturnicentric position of Iapetus is 0.0000 0590 AU. The reduction in the maximum residual is of this order and we may attribute the greater part of this reduction to the main term in the node.

The effect of this long-period term in reducing the RMS residual is much smaller since the contribution of the term to $\Delta\Omega$ varies in size as $|\sin \ell_s|$ varies over the period of the term. During a third of the period of this term, for example, $|\sin \ell_s|$ is less than a half. We should note that the interval over which the comparison is being made (50 years) is only a little more than one and a half periods of the term and so we may not expect its contribution to be properly reflected in the RMS residual. Nonetheless, the significance of this term and the other Solar terms is clear from the 25% reduction in the RMS residuals. This corresponds to approximately 0".034 as seen from 8.5 AU ; the reduction in the maximum residual corresponds to 0".106 at 8.5 AU.

We may also see the improvement in the theory by considering the graphs of the averaged residuals of the osculating elements plotted as a function of time. In Figure 1 on page 10 we show the residuals from Sinclair's (1974) theory in column (a) and those from "Harper et al (2)" (i.e. Sinclair (1974) plus Solar terms in node, inclination, apse and eccentricity plus 5:1 terms in the mean longitude derived by the method of Sinclair) in column (b). The significant periodic residual in the node has been removed, as have the periodic residuals in inclination, eccentricity and apse.

Now we consider the improvement to the theory due to the addition of the 5:1 quasi-resonance terms in the mean longitude. The residual graphs of the elements show that the periodic residuals in $\delta\lambda$ have been almost eliminated by the addition of the 5:1 terms developed in this chapter.

Comparison of the RMS and maximum theory-minus-integration residuals also shows some interesting results. The improvement in the fit of the theory to the integration may be seen by comparing the RMS residuals of Harper et al (1) i.e. without 5:1 terms, and Harper et al (2) which include these terms. The RMS residual has been reduced by 53 parts in 401 or approximately 13%. This is about half as large as the improvement produced by the addition of the Solar terms and corresponds to 0".013 as seen from 8.5 AU. Again, the reduction in the maximum residual is larger in absolute terms, though it is of the same relative size, some 17% of the maximum residual from Harper et al (1). This reduction corresponds to 0".067 seen from 8.5 AU.

The overall reduction in RMS and maximum residuals produced by addition of the Solar terms and the 5:1 Titan terms may be summarised as follows.

	RMS	Maximum
Absolute reduction	0.00000195 AU	0.00000712 AU
Relative reduction	35.9%	35.2%
Equivalent arc at 8.5 AU	0".047	0".173

The terms developed in this chapter significantly improve the closeness of the fit between the analytical theory of Iapetus and the motion of the satellite as given by Sinclair and Taylor's integration. Both RMS and maximum residuals are reduced by one third.

Sinclair and Taylor fitted Sinclair's (1974) theory to photographic observations of the satellites made between 1967 and 1982 and obtained a root-mean-square residual in the Titan-Iapetus data of 0".22. The new theory of Iapetus has an RMS residual nearly 0".05 smaller than Sinclair's theory when compared to the integration. This is a significant fraction of the $0''.22$ residual of Sinclair and Taylor and we may expect a similar improvement when the new theory is compared to the same photographic data.

Comparison of the residuals given by the Harper-Rapaport theory with those for Harper et al (1) shows the effect of adding Rapaport's 5:1 terms. The reduction in the RMS residual is less than half that produced by adding the 5:1 terms developed in this chapter, and the reduction in the maximum residual is only 3/4 as large. Rapaport's development of the 5:1 terms is incomplete as it only includes 3 of the 4 terms which have been included in this work. However, Rapaport's (1978) paper omits much detail such as analytic expressions for the coefficients in his table 2 and the values of the constants used to evaluate them. As a consequence, further critical analysis of his 5:1 terms in comparison with the derivation herein cannot be made. It is sufficient to note that the 5:1 terms developed in this chapter are to be preferred to Rapaport's terms in the form in which Rapaport presents them.

3.0 LONG-PERIOD MOTION OF THE ORBIT PLANE OF A NATURAL SATELLITE

The orbital motion of most major natural satellites in the Solar System is characterised by small perturbations from two or more sources. These may be listed :

- The gravitational attraction of other satellites orbiting the same primary
- The gravitational attraction of the Sun
- The oblateness of the primary, which causes the gravitational potential field of the primary to differ from the simple $R = -GM/r$ of a spherically symmetric body or point mass

The relative magnitudes of these effects depends in part upon the orbital distance of the satellite relative to the radius of the primary and to the orbital distance of the Sun. A satellite orbiting within a few planetary radii will be strongly affected by the oblateness of the primary ; conversely, a satellite in a large orbit, whose period is of the order of a year or more, will suffer significant solar perturbations. In either case, the motion of the satellite may also be disturbed by the presence of another massive satellite in a nearby orbit.

The secular motion of the orbit plane is of interest because it is rather sensitive to the relative sizes of the various perturbing effects. In this section we consider the secular motion of the orbit plane of a satellite which is subject to perturbations of similar magnitude from two or more sources. As an example, we consider the case of Iapetus, the ninth satellite of Saturn, which is perturbed principally by Titan and the Sun. Oblateness perturbations upon Iapetus are smaller but are not negligible.

3.1 THE DISTURBING FUNCTION

We present below the terms from the disturbing function which contain only the node and inclination of the satellite and the perturbing object (or the equator plane of the primary in the case of oblateness perturbations). Terms are given to fourth order in the sine of the inclinations and they are with respect to an arbitrary fixed reference plane. As can be seen, the three disturbing functions are very similar in form.

Solar disturbing function

$$R_s^{(2)} = -n^2a^2\chi_s \{\sin^2 I + \sin^2 I_s - 2 \sin I \cos I \sin I_s \cos I_s \cos(\Omega - \Omega_s) \}$$

[1]

$$R_s^{(4)} = -n^2a^2\chi_s \{-\tfrac{1}{2} \sin^2 I \sin^2 I_s \cos 2(\Omega - \Omega_s) - 3/2 \sin^2 I \sin^2 I_s \}$$

Oblateness disturbing function

[2]

$$R_e^{\langle 2\rangle} = -n^2a^2\chi_e \{\sin^2 I + \sin^2 I_e - 2 \sin I \cos I \sin I_e \cos I_e \cos(\Omega - \Omega_e) \}$$

$$R_e^{\langle 4\rangle} = -n^2a^2\chi_e \{-\tfrac{1}{2} \sin^2 I \sin^2 I_e \cos 2(\Omega - \Omega_e) - 3/2 \sin^2 I \sin^2 I_e \}$$

Satellite disturbing function

[3]

$$R_t^{\langle 2\rangle} = -n^2a^2\chi_t \{3 - \cos I - \cos I_t - \cos I \cos I_t - 2 \sin I \sin I_t \cos(\Omega - \Omega_t) \}$$

$$R_t^{\langle 4\rangle} = -n^2a^2\chi_t \{-2 (1 - \cos I) (1 - \cos I_t) \cos 2(\Omega - \Omega_t) \}$$

Where I = Inclination of the satellite orbit to the reference plane

Ω = Longitude of the node of the satellite orbit upon the reference plane

$R^{\langle 2\rangle}$ = first-order part of the disturbing function

$R^{\langle 4\rangle}$ = second-order part of the disturbing function

χ_s = $(3/8)\ \mu_s\ (a/a_s)^3$

χ_e = $(3/4)\ J_2\ (a_e/a)^2$

χ_t = $(1/8)\ \mu_t\ \alpha b_{3/2}^{\langle 1\rangle}$

and

M = mass of the primary

a = semi-major axis of the satellite orbit

n	= mean motion of the satellite
J_2	= dynamical form factor of the primary
μ_j	= the mass ratio of the disturbing body to the primary
α, $b_{3/2}^{(1)}$	= the ratio of the semi-major axis of the disturbing satellite to the disturbed satellite, and a Laplace coefficient.
Ω_t, I_t	= Node and inclination of the orbit of the disturbing satellite on the reference plane
a_e	= equatorial radius of the primary
Ω_e, I_e	= Node and inclination of the equator plane of the primary on the reference plane
a_s	= semi-major axis of the orbit of the Sun around the primary
Ω_s, I_s	= Node and inclination of the orbit of the Sun on the reference plane.

The perturbation in the node and inclination may be found from the Lagrange planetary equations. We need only retain the terms involving partial derivatives of R with respect to Ω and i since the other osculating elements of the satellite do not appear in the disturbing functions as given in [1], [2], [3]. Thus

[4] $$\frac{d\Omega}{dt} = \frac{1}{na^2 \sin i} \frac{\partial R}{\partial i}$$

[5] $$\frac{di}{dt} = \frac{-1}{na^2 \sin i} \frac{\partial R}{\partial \Omega}.$$

3.2 FIRST-ORDER THEORY FOR TWO DISTURBING FORCES

If there are only two perturbing effects and we neglect powers of sine inclination above the second, then the disturbing function is of the following form.

$$[6] \qquad R = -n^2a^2 \chi_1 \{I^2 + I_1{}^2 - 2 I I_1 \cos(\Omega - \Omega_1)\}$$

$$-n^2a^2 \chi_2 \{I^2 + I_2{}^2 - 2 I I_2 \cos(\Omega - \Omega_2)\}$$

Consider the perturbation in the inclination.

$$[7] \qquad dI/dt = 2n \{ \chi_1 I_1 \sin(\Omega - \Omega_1) + \chi_2 I_2 \sin(\Omega - \Omega_2) \}$$

We may choose the reference plane so that it passes through the points of intersection of the orbit planes of the disturbing bodies (or the equator plane of the primary if oblateness is one of the disturbing effects).

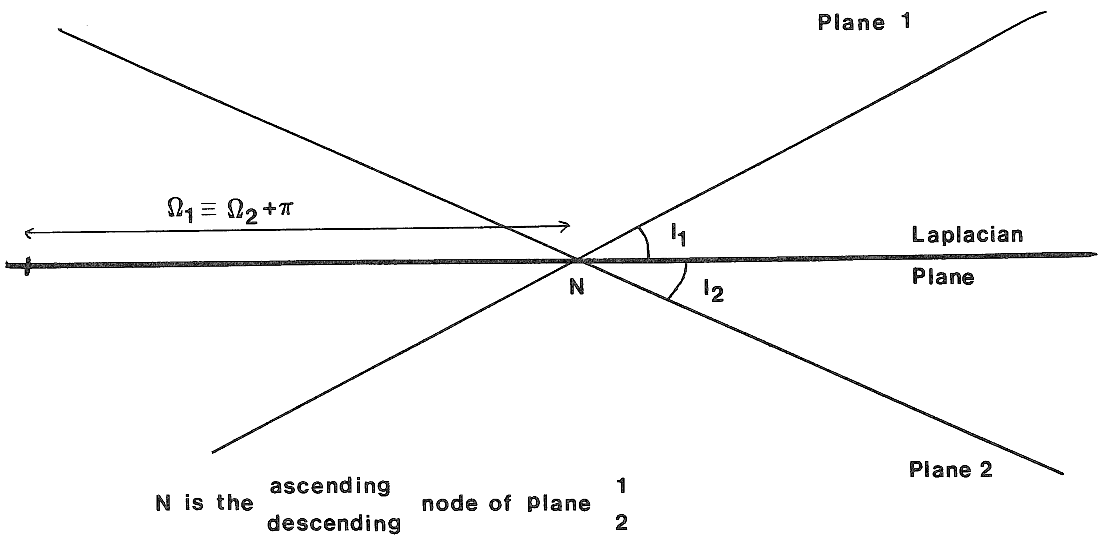


Figure 5. Laplacian plane

Then

[8] $\Omega_2 \equiv \Omega_1 - \pi$

and hence

[9] $\sin(\Omega - \Omega_2) \equiv -\sin(\Omega - \Omega_1)$

and we may write

[10] $dI/dt = 2n(\chi_1 I_1 - \chi_2 I_2)\sin(\Omega - \Omega_1).$

Clearly we may make dI/dt vanish if

$$[11] \qquad \chi_1 I_1 - \chi_2 I_2 \equiv 0.$$

This means that the inclination of the orbit of the satellite remains constant upon the plane defined in the figure. The plane is called the Laplacian plane of the satellite and it lies between the orbit planes of the disturbing bodies. Its inclination with respect to either of these planes depends upon the relative sizes of the disturbing forces, and may be found be solving [11] in conjunction with

$$[12] \qquad I_1 + I_2 = I^*$$

where I^* is the mutual inclination of the orbit planes of the disturbing bodies.

If $\underline{n}_1$ and $\underline{n}_2$ are the unit normal vectors to the orbit planes 1 and 2 respectively then the unit normal to the Laplacian plane is given by

$$[13] \qquad \underline{n}_L = (\sin I_2/\sin I^*) \ \underline{n}_1 + (\sin I_1/\sin I^*) \ \underline{n}_2$$

We may determine the motion of the node upon the Laplacian plane using [4].

[14] $d\Omega/dt = \kappa = -2n\,(\chi_1 + \chi_2)$

That is to say, the orbit precesses uniformly in a retrograde direction upon the Laplacian plane.

Let the node and inclination of the Laplacian plane upon a fixed ecliptic and equinox of epoch be Ω_L, I_L and suppose the orbit of the satellite to be inclined at a constant angle I' to the Laplacian plane. Moreover, denote by Ψ the arc of the Laplacian plane from its ascending node upon the ecliptic to the ascending node of the satellite orbit upon it, as in the accompanying figure.

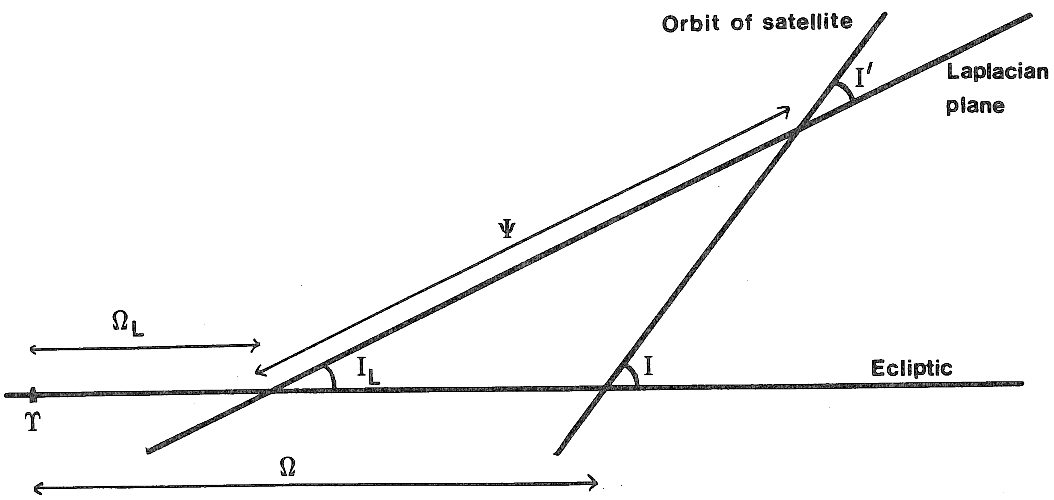


Figure 6. The Laplacian plane and the satellite orbit

Then the node and inclination of the satellite upon the ecliptic and equinox are given by

$$\begin{aligned} \sin(\Omega - \Omega_L)\sin I &= \sin I' \sin \Psi \\ \cos(\Omega - \Omega_L)\sin I &= \cos I' \sin I_L + \sin I' \cos I_L \cos \Psi \\ \cos I &= \cos I' \cos I_L - \sin I' \sin I_L \cos \Psi \end{aligned} \qquad [15]$$

where

$$\Psi = \Psi_o - \kappa t.$$

Example : Iapetus

The principal perturbing forces upon Iapetus are due to the Sun and Titan. The action of oblateness and the other inner satellites is small and is included in the disturbing function due to Titan : the equator plane of Saturn is assumed to be identical to the orbit plane of Titan. We may write

$$\chi_s = (3/8)\ \mu_s\ (a/a_s)^3$$

$$\chi_t = (1/8)\ \mu_t\ \alpha b_{3/2}^{(1)} + (3/4)\ J_2\ (a_e/a)^2 + (1/8) \sum_i \mu_i \alpha_i b_{3/2}^{(1)}(\alpha_i)$$

and we adopt the following values for the parameters of the orbit of Iapetus, the Sun and Titan, and of J_2 of Saturn, referred to the mean ecliptic and equinox of 1950.

$\Omega_s = 113^\circ.158$ $\qquad I_s = 2^\circ.4909$

$\Omega_t = 168^\circ.747$ $\qquad I_t = 27^\circ.779$

$\mu_s = 3499.4$ $\qquad a/a_s = 0.0024948$

$\mu_t = 2.383\ 10^{-4}$ $\qquad \alpha = a_t/a = 0.34314$

$3/2\ J_2 = 0.024311$ $\qquad a_e/a = 0.016853$

$n = 4^\circ.53795711$ per day

Hence $I^* = 26^\circ.445$

$\chi_s = 2.037\ 10^{-5}$

$\chi_t = 1.330\ 10^{-5} + 0.345\ 10^{-5} + 0.011\ 10^{-5} = 1.686\ 10^{-5}$

where the contribution of the oblateness and the other inner satellites to χ_t have been given explicitly (the second and third terms respectively) to show their relative sizes.

The inclination of the orbits of the Sun and Titan to the Laplacian plane may be determined to be

$$I_s = I^* \chi_t/(\chi_s + \chi_t) = 11^\circ.933$$

$$I_t = I^* \chi_s/(\chi_s + \chi_t) = 14^\circ.512.$$

The normal to the Laplacian plane is given by

$$\underline{n}_L = F_s \; \underline{n}_s + F_t \; \underline{n}_t$$

where $F_s = \sin I_t/\sin I^*$

$F_t = \sin I_s/\sin I^*$.

The components of $\underline{n}_L$ are thus (0.06471065, 0.22184702, 0.97293132) and hence the position of the pole of the Laplacian plane is

$$\Omega_L = 163^\circ.738$$

$$I_L = 13^\circ.3614$$

and the rate of motion of the orbit of Iapetus on its Laplacian plane is

$$\kappa = -11^\circ.3478 \text{ per Julian century.}$$

We may determine the position of the orbit of Iapetus upon its Laplacian plane by comparison with mean elements given by Struve (1933) and Sinclair (1974). The mean node and inclination at several epochs are presented in the table below, referred to the mean ecliptic and equinox of 1950. The values obtained directly from observations include long period perturbations due to the Sun. These terms must be subtracted in order to determine the underlying secular variations of the node and inclination. Struve subtracted one Solar term in forming his mean points but the theory of the Solar perturbations upon Iapetus has been revised by Sinclair (1974) and by Harper et al (in submission). The points from Struve have been corrected by subtraction of the periodic perturbations given by Sinclair (1974) and Harper et al (in submission) ; the point from Sinclair (1974) has been corrected by subtracting the periodic perturbation in Harper et al.

Date	Node	Inclination
1787.70	146°.50627	19°.26139
1832.50	145°.05501	18°.85991
1857.50	144°.32894	18°.70378
1876.70	143°.39184	18°.54219
1880.20	143°.35909	18°.54550
1885.60	143°.10195	18°.46443
1917.20	141°.96303	18°.15884
1918.20	141°.91871	18°.15050
1926.40	141°.55608	18°.06129
1927.40	141°.57161	18°.05516
1973.00	139°.89829	17°.56719

We solve for Ω and I in equations [15] and for μ_T. The values obtained are

$$\Omega = 143^\circ.084 \pm 0^\circ.040 \text{ at the epoch } 1885.25$$

$$I = 18^\circ.449 \pm 0^\circ.013$$

$$\mu_T = (2.333 \pm 0.063)\times 10^{-4}.$$

The residuals $\sin \iota\, \Delta\Omega$ and $\Delta\iota$ are given in the following table.

Date	$\sin \iota \; \Delta\Omega$	$\Delta\iota$
1787.70	$-0^\circ.0997$	$-0^\circ.0139$
1832.50	$-0^\circ.0096$	$-0^\circ.0576$
1857.50	$+0^\circ.0611$	$+0^\circ.0003$
1876.70	$-0^\circ.0089$	$+0^\circ.0100$
1880.20	$+0^\circ.0220$	$+0^\circ.0452$
1885.60	$+0^\circ.0039$	$+0^\circ.0136$
1917.20	$+0^\circ.0102$	$+0^\circ.0071$
1918.20	$+0^\circ.0077$	$+0^\circ.0085$
1926.40	$-0^\circ.0128$	$-0^\circ.0004$
1927.40	$+0^\circ.0032$	$+0^\circ.0033$
1973.00	$-0^\circ.0134$	$-0^\circ.0192$
Root-mean-square	$0^\circ.0368$	$0^\circ.0240$

Inclination of the orbit of Iapetus to its Laplacian plane = $7^\circ\ 33'.0$

3.3 FIRST-ORDER THEORY WITH MORE THAN TWO DISTURBING FORCES

When there are more than two significant disturbing forces, the formulation of the previous section is less tractable. Equation [7] has three or more terms on the right-hand side and cannot be readily transformed into the form of equation [10]. We elect to use a different approach by introducing new variables p and q defined by

$$p = \sin I \sin \Omega$$

[16]

$$q = \sin I \cos \Omega.$$

Lagrange's planetary equations now become

$$na^2\, dp/dt = +\cos I\, \partial R/\partial q \simeq +\partial R/\partial q$$

[17]

$$na^2\, dq/dt = -\cos I\, \partial R/\partial p \simeq -\partial R/\partial p.$$

This form of the equations was used by Tisserand (1892). We use the approximate forms, assuming $\cos I \simeq 1$.

The first-order disturbing functions $R_s^{\langle 2 \rangle}$, $R_t^{\langle 2 \rangle}$, $R_e^{\langle 2 \rangle}$ may each be written in the form

[18] $$R_i^{\langle 2 \rangle} = -n^2a^2\, \chi_i\, \{p^2 + q^2 - 2\alpha_i p - 2\beta_i q + \alpha_i^2 + \beta_i^2\}.$$

The total disturbing function due to N perturbing forces may be written

[19] $$R = -n^2a^2K\, \{p^2 + q^2 - 2Ap - 2Bq + C\}$$

where

$$K = \sum_i \chi_i$$

[20]

$$A = \left(\sum_i \chi_i \alpha_i \right)/K$$

$$B = \left(\sum_i \chi_i \beta_i \right)/K$$

$$C = \left(\sum_i \chi_i \left(\alpha_i^2 + \beta_i^2\right) \right)/K.$$

Tisserand (1892) showed that the disturbing function has the following property when restricted to the secular terms described in this work.

[21] $$\frac{dR}{dt} = \frac{dp}{dt}\frac{\partial R}{\partial p} + \frac{dq}{dt}\frac{\partial R}{\partial q} = 0$$

That is to say

[22] $$R = \text{constant}$$

hence

[23] $$p^2 + q^2 - 2Ap - 2Bq + C = \text{constant}.$$

This is the equation of a circle

$$[24] \qquad (p - p_o)^2 + (q - q_o)^2 = r^2$$

where

$$[25] \qquad p_o = A \quad , \quad q_o = B$$

define the centre of the circle in the pq-plane. Referring back to the previous section, it is evident that (p_o, q_o) is the position of the pole of the Laplacian plane of the orbit. The orbit maintains a constant inclination to the Laplacian plane given by $r = \sin I$ where r^2 is the constant right-hand side of equation [24]. If Ω_L, I_L denote the node and inclination of the Laplacian plane in the fixed reference system then

$$[26] \qquad \begin{aligned} p_o &= \sin I_L \sin \Omega_L \\ q_o &= \sin I_L \cos \Omega_L . \end{aligned}$$

We may regard p_o, q_o as the coordinates of the 'centroid' or 'weighted mean' of the coordinates of the poles of the disturbing forces. The 'weight' of each pole (α_i, β_i) is the dimensionless coefficient χ_i in equation [18].

If we choose the coordinate system so that the reference plane is the Laplacian plane then $I_L = 0$ and hence $p_o = q_o = 0$. We then have

[27] $$R = -n^2a^2 K (p^2 + q^2 + C')$$

$$= -n^2a^2 K U.$$

Applying equations [17] we have

$$dp/dt = - nK \partial U/\partial q = -2n K q$$

[28]

$$dq/dt = + nK \partial U/\partial p = +2n K p.$$

The solution to these equations may be written

$$p = r \sin (\kappa t - \phi)$$

[29]

$$q = r \cos (\kappa t - \phi)$$

where

[30] $$\kappa = -2n K$$

and r, ϕ are to be determined from observations. ϕ is the node of the orbit plane upon the Laplacian plane at time $t = 0$ and r is as in equation [24].

The rate of precession of the orbit upon the Laplacian plane is 2K times the mean motion of the satellite - cf. equation [14].

3.4 SECOND-ORDER THEORY WITH AN ARBITRARY NUMBER OF DISTURBING FORCES

We now consider the disturbing functions [1], [2], [3] up to fourth order in the inclinations, i.e. including terms such as $R_s^{(4)}$, $R_t^{(4)}$, $R_e^{(4)}$. Using the notation of the previous section together with

[31]
$$D = \Big(\sum_i \chi_i \alpha_i \beta_i \Big)/K$$
$$\gamma = \Big(\sum_i \chi_i k_i (\alpha_i^2 + \beta_i^2) \Big)/K$$
$$\varepsilon = \tfrac{1}{2} \Big(\sum_i \chi_i (\beta_i^2 - \alpha_i^2) \Big)/K$$

we have

[32]
$$R = - n^2a^2K \{(1 - \gamma + \varepsilon)p^2 + (1 - \gamma - \varepsilon)q^2 - 2Ap - 2Bq - 2Dpq + C\}.$$

This differs from equation [19] in two respects : (i) the coefficients of p^2 and q^2 are no longer equal, and (ii) there is now a second-order cross term in pq. Equations [22] and [32] imply that the pole of the orbit follows an ellipse. The centre of the ellipse (p_o, q_o) may be found by substituting $p = p_o + x$, $q = q_o + y$ into [32] and making the resulting coefficients of x and y equal to zero. We find

[33]
$$p_o = (BD + A(1 - \gamma - \varepsilon))/\Gamma^2$$
$$q_o = (AD + B(1 - \gamma + \varepsilon))/\Gamma^2$$

where

$$\Gamma^2 = (1 - \gamma)^2 - \varepsilon^2 - D^2.$$

This is the pole of the Laplacian plane of the orbit - cf. equation 25. The disturbing function becomes

[34] $$R = - n^2a^2K\{ (1 - \gamma - \varepsilon)x^2 + (1 - \gamma + \varepsilon)y^2 - 2Dxy + G\}$$

where G is a new constant. Then

[35] $$dp/dt = dx/dt = - 2nK \{ (1 - \gamma + \varepsilon)y - Dx\}$$
$$dq/dt = dy/dt = + 2nK \{ (1 - \gamma - \varepsilon)x - Dy\}.$$

These equations admit the general solution

[36] $$x = x_c \cos \kappa t + x_s \sin \kappa t$$
$$y = y_c \cos \kappa t + y_s \sin \kappa t$$

where the rate of precession κ is given by

[37] $$\kappa = - 2nk \Gamma.$$

The four coefficients x_c, y_c, x_s, y_s are not independent. If we choose x_c, y_c to be arbitrary constants determined from observations then x_s, y_s are

$$x_s = \{+ Dx_c - (1 - \gamma - \varepsilon)y_c\}/\Gamma$$

[38]

$$y_s = \{- Dy_c - (1 - \gamma + \varepsilon)x_c\}/\Gamma .$$

Example : Iapetus

We may refine the theory of the motion of the orbit plane of Iapetus by re-calculating the rate of precession λ and the position of the pole of the Laplacian plane. Using the previous position of the Laplacian plane as a reference plane we may define a triad of orthonormal vectors $\underline{N}$, $\underline{M}$, $\underline{W}$ such that $\underline{N}$ is in the line of intersection of the Laplacian plane with the ecliptic, at the ascending node; $\underline{M}$ is in the Laplacian plane 90° from $\underline{N}$ and $\underline{W}$ is normal to the Laplacian plane.

Let $\Omega_L{}^0$, $I_L{}^0$ be the node and inclination of the Laplacian plane obtained in the previous section and now to be used as a first approximation. Then

$$\underline{N}_L{}^0 = \begin{cases} + \cos \Omega_L{}^0 \\ + \sin \Omega_L{}^0 \\ 0 \end{cases}$$

$$\underline{M}_L{}^0 = \begin{cases} - \sin \Omega_L{}^0 \cos I_L{}^0 \\ + \cos \Omega_L{}^0 \cos I_L{}^0 \\ + \sin I_L{}^0 \end{cases}$$

$$\underline{W}_L{}^0 = \begin{cases} + \sin \Omega_L{}^0 \sin I_L{}^0 \\ - \cos \Omega_L{}^0 \sin I_L{}^0 \\ + \cos I_L{}^0 \end{cases}$$

Let Ω_i, I_i be the node and inclination of the orbit of the disturbing body referred to the ecliptic and equinox of 1950 and $\underline{n}_i$ be the unit normal to that plane. Then

$$\alpha_i = \underline{n}_i \bullet \underline{N}_L{}^0$$

$$\beta_i = \underline{n}_i \bullet \underline{M}_L{}^0$$

We tabulate below the values of α, β and χ for the Sun, Titan and the oblateness of Saturn. We adopt the following parameters (cf. previous example.)

Sun : $\Omega_s = 113°.158$ $I_s = 2°.4909$

$\chi_s = 2.03726\ 10^{-5}$

Oblateness : $\Omega_e = 168°.710$ $I_e = 28°.1410$

$\chi_e = 0.34525\ 10^{-5}$

Titan : $\Omega_t = 168°.747$ $I_t = 27°.7790$

$\chi_t/\mu_t = 0.0558215$

$\mu_t = 2.3829\ 10^{-4}$ $\chi_t = 1.33017\ 10^{-5}$

Mean motion of Iapetus = $4°.53795711$ per day

First-order position of the Laplacian plane :

$\Omega_L{}^0 = 163°.723$ $I_L{}^0 = 13°.3670$

Body	α	β	$\chi \times 10^5$
Sun	-0.033491	+0.199849	2.0373
Titan	+0.042086	-0.251164	1.3217
Oblateness	+0.042287	-0.257276	0.3452

We obtain the following results upon fitting to the data.

Rate of precession of the orbit upon the Laplacian plane :

$\kappa = -\ 12°.2793$ per Julian century

(cf. $-11°.3478$ in the first-order theory)

Position of the pole of the Laplacian plane :

$$p_o = +0.0005837$$

$$q_o = -0.0039601$$

$$\Omega_L = 163^\circ.711$$

$$I_L = 13^\circ.3497$$

Solution of the equations for p and q :

$$p = p_o \quad -0.111288 \cos \kappa t \quad +0.068650 \sin \kappa t$$

$$q = q_o \quad -0.069426 \cos \kappa t \quad -0.114809 \sin \kappa t$$

We may deduce the maximum and minimum values of the inclination of the orbit of Iapetus to the Laplacian plane :

$$I_{maximum} = 7^\circ\ 43'.0$$

$$I_{minimum} = 7^\circ\ 30'.5$$

The range of inclination is thus $7^\circ\ 36'.8 \ \pm 6'.3$

(cf. $7^\circ\ 33'.0$ in the first-order case)

The residuals $\sin \iota\ \Delta\Omega$ and $\Delta\iota$ are given in the following table.

Date	$\sin \iota \; \Delta\Omega$	$\Delta\iota$
1787.70	$-0^\circ.1627$	$-0^\circ.0435$
1832.50	$-0^\circ.0456$	$-0^\circ.0744$
1857.50	$+0^\circ.0406$	$-0^\circ.0084$
1876.70	$-0^\circ.0176$	$+0^\circ.0078$
1880.20	$+0^\circ.0155$	$+0^\circ.0441$
1885.60	$+0^\circ.0008$	$+0^\circ.0145$
1917.20	$+0^\circ.0270$	$+0^\circ.0194$
1918.20	$+0^\circ.0252$	$+0^\circ.0212$
1926.40	$+0^\circ.0100$	$+0^\circ.0154$
1927.40	$+0^\circ.0266$	$+0^\circ.0195$
1973.00	$-0^\circ.0391$	$-0^\circ.0152$

Root-mean-square $\quad 0^\circ.0560 \quad 0^\circ.0322$

The values of the node and inclination of the orbit of Iapetus at the epoch 1885.25 and the mass of Titan are found to be :

$$\Omega = 143^\circ.093 \pm 0^\circ.045$$

$$I = 18^\circ.448 \pm 0^\circ.014$$

$$\mu_T = (2.252 \pm 0.067)\times 10^{-4}$$

3.5 DISCUSSION OF RESULTS : DETERMINATION OF THE MASS OF TITAN

Previous authors have used the motion of the orbit plane of Iapetus to determine the mass of Titan. As we have shown, the rate of precession of the orbit plane upon its Laplacian plane is directly dependent upon the mass of Titan (cf. equations [14], [30], [37] and the expansions for χ_s, χ_e, χ_t given after equation [3]).

We give below a table of values from Jeffreys (1953), Sinclair (1974) and this work, plus the values obtained by Sinclair and Taylor (1985) from an analysis of the orbits of Titan, Hyperion and Iapetus by numerical integration, by Tyler et al (1981) from analysis of Voyager 1 radio-tracking data and by Message from the motion of Hyperion where value (a) is a weighted mean of values obtained from individual terms in the theory of Hyperion and value (b) is a least-squares solution.

Source	$\mu_T \times 10^4$
Jeffreys	2.412 ± 0.018
Jeffreys (from Iapetus)	2.357 ± 0.052
Sinclair	2.422 ± 0.031
1^{st}-order Laplacian plane	2.333 ± 0.063
2^{nd}-order Laplacian plane	2.252 ± 0.067
Sinclair and Taylor	2.36777 ± 0.00055
Tyler et al	2.3664 ± 0.0008
Message (a)	2.3648 ± 0.0055
Message (b)	2.3677 ± 0.0004

These values may be visualised in the following figure.

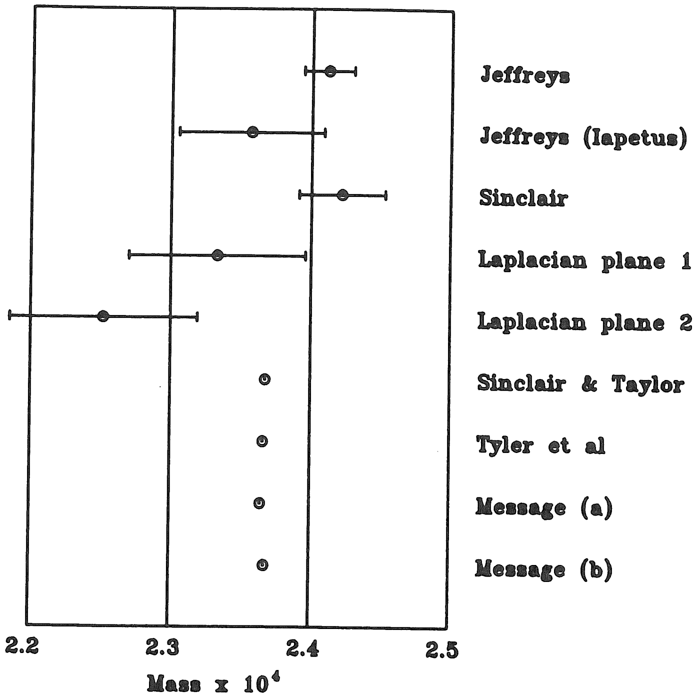


Figure 7. Determination of the mass of Titan

The values of μ_T obtained here are consistent with those obtained by other authors, notably Sinclair and Taylor, Tyler et al and Jeffreys' determination from the motion of the orbit plane of Iapetus. The value from the first-order theory is in better agreement with other values than that from the second-order theory. In addition, the first-order theory gives a better fit to the data since its root-mean-square residuals in $\sin \iota \, \Delta\Omega$ and $\Delta\iota$ are smaller by some 30% than those from the second-order theory. We are probably justified, therefore, in preferring the value of μ_T from the first-order theory as the more reliable.

The main difference between the two theories, particularly when fitting to data over a period of two centuries, lies in the precession rate κ. The second-order theory has a rate some 8% larger than the first-order theory due to the factor Γ and we may expect the two models to diverge over long periods of time. The difference amounts to $0^\circ.931$ per Julian century or $1^\circ.73$ over the span of the data. This will not be manifested in the calculated values of the node and inclination however, since the parameters of the model will change to accommodate different precession rates as part of the least-squares fitting process. It may be that the mass of Titan from the second-order theory is low because the fitting process tried to reduce the precession rate : the precession rate implied (in the second-order expression for κ) by a value of $\mu_T = 2.35\ 10^{-4}$ (say) may in fact be too large than the rate represented by the observed values of node and inclination. However, the data only covers 200 years of a 3000 year precession and the rate is not well-determined anyway, as the large standard error of μ_T attests. The mass of Titan may be obtained more accurately by other means.

It is important to note that the fourth-order terms in the disturbing functions given at the start of this chapter do not contain the fourth power of the inclination of the satellite. They are fourth-order only in the sense that the powers of the various inclinations in each term sum to 4. This allows us to retain the more tractable form of the equations for dp/dt and dq/dt. Inclusion of the fourth power of the satellite's inclination would introduce terms in p^3 and q^3 into the expressions for dq/dt and dp/dt respectively. In the case of mutual satellite terms we would also need to expand the planetary disturbing function to include terms whose Laplace coefficients have fractional subscript 3/2.

4.0 PREPARATION OF VISUAL OBSERVATIONS

In this chapter we develop the theory required to compare the observations of the satellites of Saturn with any dynamical theory. The task falls into three main parts :

1. Conversion of the time scales of the data into Universal Time and then into Ephemeris Time, which is the time scale in which the dynamical theories are expressed. We also consider one or two other minor points relating to time- scales.

2. The calculations required to obtain a computed topocentric position of a satellite from its Saturnicentric theory. Such a topocentric position must be directly comparable to observations and so it must incorporate a number of important physical effects in addition to the simple translation of the origin from Saturn to the observer.

3. In order to enter a differential correction process on the parameters of the theory it is necessary to calculate the partial derivatives of the observed quantities with respect to the Saturnicentric coordinates of the satellite.

The solutions to these problems are covered in the following sections.

4.1 SYSTEMS OF TIME MEASUREMENT

Several time scales are used in the data sets. In each case we must devise a procedure for converting the given time of each observation into Universal Time. The only case where any complication might arise is in the treatment of Local Apparent Sidereal Time, although some care is also required when dealing with Mean Astronomical Time systems (such as WMAT). The method of conversion from LST to UT and from WMAT to UT is given in the next two sub-sections. As will be shown in a subsequent section, we require the Local Sidereal Time of each observation regardless of the time scale used for the observation itself and so a method for calculating LST from UT is also given.

The dynamical theories of the satellites of Saturn have as their independent variable yet another time scale, known as Ephemeris Time (E.T.). The significance of this concept is rather important and will be discussed in a further sub-section where we also consider the effect of errors in the time argument as they are reflected in the observed positions of the satellites.

The final sub-section deals with the correction to the 'observed' time to allow for the 'light-time' delay. The practical method for making this correction involves an iteration to determine the topocentric position of Saturn and is thus also relevant to the section on reference frames.

4.1.1 CONVERSION OF WMAT TO UNIVERSAL TIME

Washington Mean Astronomical Time is a solar time scale which runs exactly 12 hours behind Washington Mean Civil Time. An astronomical day begins at midday on the corresponding civil day : thus the astronomical day 1875 February 7 begins at mean midday on the civil day 1875 February 7 and ends at mean midday on the civil day 1875 February 8.

The conversion from Washington Mean Astronomical Time to Universal Time is carried out in the following way :

Example

Given WMAT = 1875 February 7 10h 14m 23s

(1) Add 12h to get Washington Mean Civil Time

WMCT = 1875 February 7 22h 14m 23s

(2) Add the longitude of the observatory, expressed in hours, minutes and seconds and measured in a positive sense westwards.

The longitude of USNO is +5h 08m 15.71s

Thus the Universal Time is 1875 February 7 22h 14m 23s
+ 5h 08m 16s

= 1875 February 8   3h 22m 39s

So 1875 February 7 10h 14m 23s WMAT corresponds to 1875 February 8 3h 22m 39s Universal Time.

4.1.2 CONVERSION OF LST TO UNIVERSAL TIME

All observations published by H. Struve and G. Struve are measured using Local (Apparent) Sidereal Time. The conversion from LST to UT is illustrated by the following example, taken from G. Struve (Heft 2, page 20).

Example

Given Local (Apparent) Sidereal Time = 1916 January 11d 5h 10m 57s and λ (Babelsberg) = - 0h 52m 25.49s West

(1) This instant falls somewhere during the Astronomical Day of January 11 i.e. between about January 11d 18h UT and January 12d 6h UT.

(2) Strictly speaking, we should apply the correction for nutation (Equation of the Equinoxes) at the start of the calculation, but it is so small that it can be applied to the final derived UT. We shall neglect it in this example.

(3) Greenwich Sidereal Time = Local Sidereal Time + Longitude West

= 5h 10m 57s - 0h 52m 25s

= 4h 18m 32s

(4) Julian Day Number for 1916 January 11d 12h UT = 2420874.0

d = JD - 2415020.0 = 5854.0

T_u = d/36525 = 0.160273785

$$GMST_o = 23925^s.836 + 8640184^s.542 \; T_u + 0^s.0929 \; T_u^2$$

$$= 7^h \; 18^m \; 40^s.9$$

GMST at 1916 January 11d 12h UT = $19^h \; 18^m \; 40^s.9$

(5) Elapsed interval of Sidereal Time

$$\Delta(ST) = (24^h +)\ 4^h\ 18^m\ 32^s - 19^h\ 18^m\ 40^s.9$$

$$= 8^h\ 59^m\ 51^s$$

(6) Elapsed interval of Universal Time

$$\Delta(UT) = \Delta(ST) \times \alpha \qquad \alpha = 0.99726\ 95664$$

$$= 8^h\ 58^m\ 23^s$$

Thus the time is JD 2420874 + $8^h\ 58^m\ 23^s$ = 2420874.37388

4.1.3 CONVERSION OF UT TO LOCAL SIDEREAL TIME

We require the Local Apparent Sidereal Time at the instant of each observation in order to calculate the topocentric correction vector i.e. the topocentric position vector of the geocentre referred to the True Equator and Equinox of Date. For those observations where the LST is not explicitly given (that is, all data except that published by Struve father and son) we must calculate the Local Sidereal Time from the Universal Time of the observation. The procedure is described in the Explanatory Supplement and also in standard works on positional and observational astronomy such as Smart, so we do not provide details here.

4.1.4 UNIVERSAL TIME AND EPHEMERIS TIME

All of the observations of the satellites of Saturn are measured using time scales which are defined by the rotation of the Earth on its axis with respect to the Mean Equator and Equinox of date. The fundamental time scale so defined is called Universal Time.

The dynamical theories describing the motion of the satellites have a different time scale as their independent argument. This is Ephemeris Time and it is defined (and determined) by observations of the Sun, Moon and planets. That is to say, ET is defined by the orbital motions of several bodies in the Solar System, principally the Moon. Hence ET is defined by a different physical system to that defining UT. This difference is manifested by the behaviour of the quantity $\Delta T = ET - UT$. ΔT is non-zero and it is not constant : at the present time it has a value of about 60s and is increasing by approximately 1 second per year. A graph of ΔT together with a table of values from 1621 to 1972 is given in the Explanatory Supplement (pp 90 - 91).

In order to obtain a valid argument for our chosen dynamical theory, whether it is analytic or numeric, we must add ΔT to the UT of each observation so that we shall have a time expressed in the (dynamical) ET scale.

4.1.5 LIGHT-TIME AND THE TOPOCENTRIC POSITION OF SATURN

So far, all conversion and correction operations have been carried out on the topocentric time of the observation. We have the Ephemeris Time at which the observation was made on the Earth. However, we are observing a system which is some 8 or 9 AU distant and hence because the speed of light is finite, we see the satellite system as it was about 70 minutes ago. This time-lag is called light-time : it is the time taken for light from the system to reach the observer and it must be subtracted from the ET of the observation in order to yield the time argument with which we enter the dynamical theory.

The light-time is determined by an iterative process. It is assumed that we know the heliocentric position vector of the observer at the ET of the observation and that we possess a heliocentric theory of the motion of Saturn. The procedure may be represented as the following algorithm :

(1) Let the Ephemeris Time of the observation at the Earth be t and the topocentric position vector of the centre of the Sun at this instant be $\underline{R}$; as a first approximation set the light-time τ to zero : $\tau = 0$.

(2) Enter the heliocentric theory of Saturn with time argument $t - \tau$. Write the position vector (referred to the True Equator and Equinox of Date) as $\underline{r} = \underline{r}(t - \tau)$.

(3) Calculate the light-time from

$$[1] \qquad \tau = \beta |\underline{R} + \underline{r}|$$

where β is the light-time for unit distance, 5.77559 10^{-3} days per AU.

(4) Repeat steps (2) and (3) until successive values of τ converge. This is the light-time to be subtracted from t in order to obtain the time argument for the dynamical models. The vector $\underline{R} + \underline{r}$ gives the topocentric position vector of Saturn for the observation.

4.1.6 ERROR IN THE TIME

A number of simplifying assumptions have been made when calculating the time argument for each observation. Since we are describing a dynamical system, an error in the time will cause a corresponding error in the Saturnicentric positions of the satellites and hence also in their positions as seen from the Earth. In this section we consider the size of such errors and their effect upon the reduction of the observations.

The errors are of two types :

(1) Systematic and/or random errors in the calculation of the time of the observation i.e. at the Earth before correction for light-time. Such errors may result from several sources.

- Error in the time given in the original reference.
- Error in the longitude adopted to convert LST and WMAT to UT.
- Error in the value adopted for ΔT.
- Error caused by neglecting the equation of the equinoxes ($\Delta\psi.\cos\varepsilon$) in the conversion of Local Apparent Sidereal Time to UT.
- Rounding error in the Julian Date ($0^d.00001 \simeq 0^s.86$).

The effect of random errors may be determined to first order by assuming that the satellite is moving in a circular orbit. We may calculate the distance moved by a satellite in its orbit in a given small time interval Δt. The distance is

[2] $$\Delta x = a.n.\Delta t$$

where a = semi-major axis of the orbit

n = mean motion of the satellite.

On the further assumption that the satellite system is viewed from an average distance of 8.5 AU then we can calculate the arc subtended by a distance Δx as seen from the Earth. This arc $\Delta\phi$ is the maximum error in the observed position of each satellite relative to Saturn due to an error Δt in the time argument of the theory. We may write

[3] $$\Delta\phi = \Delta x/8.5.$$

The arc $\Delta\phi$ corresponding to Δt = 1 second is given for each satellite in the table below. The error Δt required to produce a $\Delta\phi$ of 0.1 arc seconds is also given, expressed in days and in seconds.

The effect of an error in the time

Satellite	$\Delta\phi/''$ (for Δt=1s)	Δt (for $\Delta\phi=0''.1$)	
		days	seconds
Mimas	0.0023	0.000498	43.0
Enceladus	0.0021	0.000565	48.8
Tethys	0.0018	0.000628	54.3
Dione	0.0016	0.000711	61.5
Rhea	0.0014	0.000841	72.6
Titan	0.00090	0.00128	111
Hyperion	0.00082	0.00141	122
Iapetus	0.00053	0.00219	189

Clearly, for the outer satellites an error in the time of up to 10 seconds will not alter the observed position by more than 0".01. The observations are rather insensitive to small errors in the time.

(2) Error in the light time : we have assumed that the light time from the observer to each of the satellites is equal to that from the observer to the centre of Saturn. This is equivalent to the assumption that the satellites and Saturn are all equidistant from the observer. Evidently, depending upon the position in its orbit, each satellite will be closer or further than the centre of Saturn and so will require a slightly different light time. In a completely rigorous calculation we would use the iterative process described in the previous section upon each satellite in order to obtain individual light times, but we will show that the simplifying assumption does not introduce serious errors.

Consider the following diagram, which represents the orbit of a satellite (taken to be circular to first order).

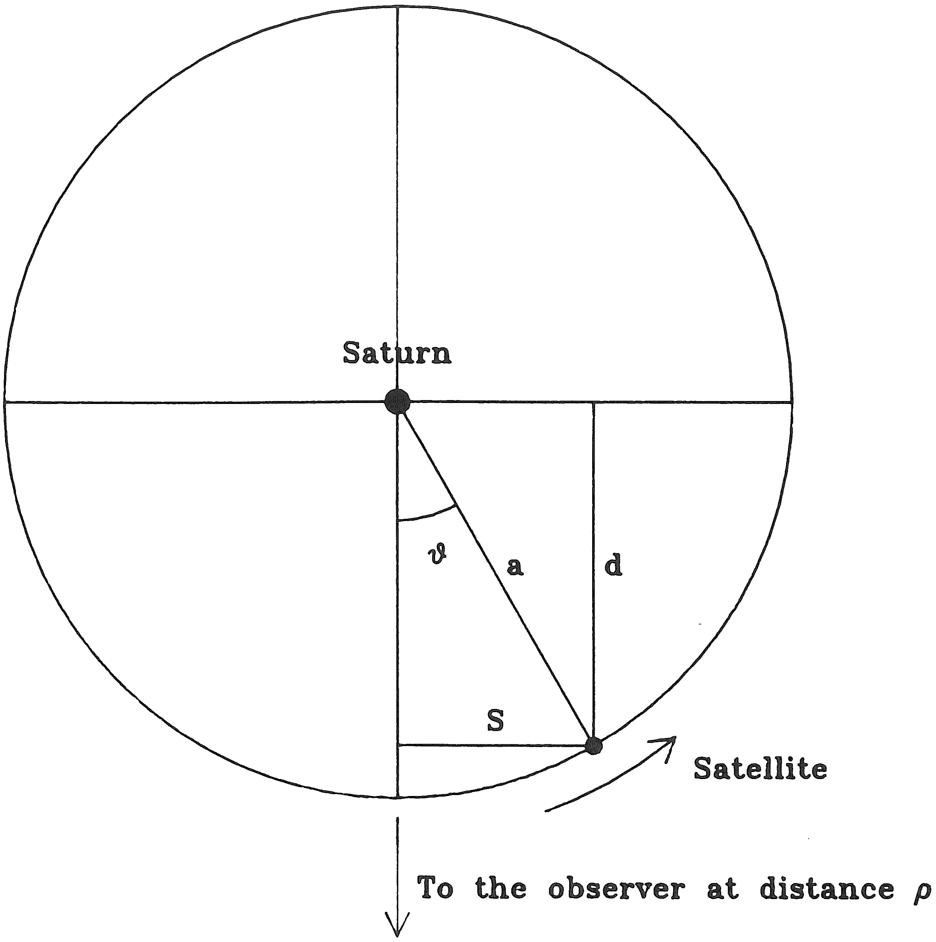


Figure 8. Light time error

$\Delta s = a.\cos\vartheta.\Delta\vartheta$ $\Delta\vartheta = n.\Delta t$

$\Delta t = -d/c$ $\Delta\psi = \Delta s/\rho$

Thus

[4] $\Delta\psi = -(na^2/\rho c)\cos^2\vartheta.$

The following table gives the value of $x = na^2/\rho c$ for each satellite. This is the maximum value of $\Delta\psi$. We assume $\rho = 8.5$ AU and $c = 173.14$ AU/day. This gives $x = na^2/1472 = 880.4\ a^2/P$ where P is the orbital period in days.

Light-time error

Satellite	x/arc-seconds
Mimas	0.0014
Enceladus	0.0016
Tethys	0.0018
Dione	0.0020
Rhea	0.0024
Titan	0.0037
Hyperion	0.0041
Iapetus	0.0063

We see that the effect of the error in the light-time is quite negligible for all the satellites.

4.2 COORDINATE SYSTEMS AND REFERENCE FRAMES

Each observation of the position of one satellite relative to another is made in a topocentric coordinate system which is unique and peculiar to the time and place of the observation. It is based upon the True Equator and Equinox of date but it includes the effects of aberration and of refraction insofar as they alter the relative positions of the two satellites. We may call this the 'O - frame'.

The dynamical theory describing the motions of the satellites is, by contrast, set in a fixed coordinate system (the 'I - frame') based upon <u>either</u> the Mean Equator and Equinox of the epoch B1950.0 <u>or</u> upon the equator plane of Saturn and its intersection with the Earth's Mean Equator of B1950.0. In order to compare the dynamical theory with observations it is necessary to apply a transformation to convert the position of the satellite in the I-frame into an observed position in the O-frame. The transformation consists of four stages :

(1) A rotation to convert from the Mean Equator and Equinox of B1950.0 (or Saturn's equator and the Earth's Mean Equator of B1950.0) to the True Equator and Equinox of Date.

After this stage we have Saturnicentric positions referred to the True Equator and Equinox of Date. The coordinate axes are now parallel to those used to express the topocentric position vector of Saturn.

(2) A translation to shift the origin from the centre of Saturn to the observer (i.e. the topocentre). This translation is carried out by adding the topocentric position vector of the centre of Saturn ($\underline{R}$) to the Sa-

turnicentric position vector of each satellite ($\underline{r}$) as shown in the following diagram.

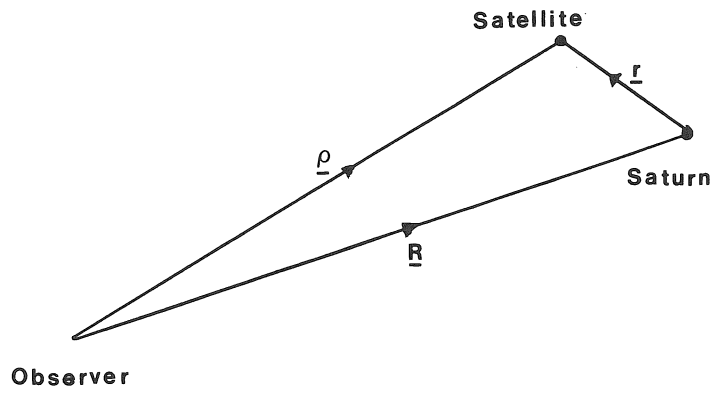


Figure 9. Topocentric position

We now have the topocentric position of each satellite referred to the True Equator and Equinox of date. However, it has been assumed that the satellite is at rest with respect to the observer. This is not so. Hence we require :

(3) A correction to allow for the velocity of the satellite relative to the observer. This correction is actually a Lorentz transformation in a simplified form and is called **aberration**.

(4) Finally, we must make allowance for the optical properties of the Earth's atmosphere through which, of course, all observation are made. The elevation of celestial bodies above the horizon is affected by atmospheric refraction. The size of the refraction depends upon the elevation itself.

Each of these stages are described fully in the following sections.

4.2.1 ROTATION TO THE TRUE EQUATOR AND EQUINOX OF DATE

We may assume that the Saturnicentric positions of the satellites given by the dynamical model are referred to the equator of Saturn and its intersection with the Earth's Mean Equator of B1950.0. This is the system used in the numeric integration program 'Titan' and it is defined thus :

1. The coordinate system has the centre of Saturn as its origin.

2. The equator plane of Saturn (i.e. its plane of axial symmetry) is the xy-plane of the coordinate system. This, together with (1), implies that the z-axis of the system is identical to Saturn's axis of rotational symmetry.

3. The direction of the x-axis of the system is defined by the ascending node of the equator plane of Saturn upon the Earth's Mean Equator of B1950.0

If we define

N = the Right Ascension of the ascending node of the equator plane of Saturn on the Earth's Mean Equator of B1950.0, referred to the Mean Equinox of B1950.0

I = the inclination of the equator plane of Saturn to the Earth's Mean Equator of B1950.0

then we may immediately write a transformation matrix to convert a vector from the reference frame defined above (the I-frame) to the Mean Equator and Equinox of B1950.0.

Let $(X, Y, Z)_I$ be the components of any vector referred to the integration frame and let $(X, Y, Z)_{B1950.0}$ be the components of the same vector referred to the Mean Equator and Equinox of B1950.0.

Then we may write

[5]
$$\begin{aligned} X_{1950.0} &= R_{11}X_I + R_{21}Y_I + R_{31}Z_I \\ Y_{1950.0} &= R_{12}X_I + R_{22}Y_I + R_{32}Z_I \\ Z_{1950.0} &= R_{13}X_I + R_{23}Y_I + R_{33}Z_I \end{aligned}$$

where the components of the matrix R are defined thus

$$[6] \qquad \begin{aligned} R_{11} &= + \cos N &&= -0.6215608247 \\ R_{12} &= - \sin N \cos I &&= -0.7780537554 \\ R_{13} &= + \sin N \sin I &&= +0.0910741178 \end{aligned}$$

$$[7] \qquad \begin{aligned} R_{21} &= + \sin N &&= +0.7780537554 \\ R_{22} &= + \cos N \cos I &&= -0.6173459053 \\ R_{23} &= - \cos N \sin I &&= +0.0722626596 \end{aligned}$$

$$[8] \qquad \begin{aligned} R_{31} &= 0 &&= +0.0000000000 \\ R_{32} &= + \sin I &&= +0.1162599970 \\ R_{33} &= + \cos I &&= +0.9932188143 \end{aligned}$$

The numerical values of the coefficients are those used in the reduction of the observations and are based upon the following assumed values for I and N.

$$N = 8^h\ 33^m\ 43^s$$

$$I = 6^\circ\ 40'\ 35''$$

The second operation we must perform upon the position is the transformation from the Mean Equator and Equinox of B1950.0 to the True Equator and Equinox of Date. This transformation is a rotation and has two parts.

(1) Precession, which is the transformation from the Mean Equator and Equinox of B1950.0 to the Mean Equator and Equinox of Date.

(2) Nutation, which is the transformation from the Mean Equator and Equinox of Date to the True Equator and Equinox of Date.

Both transformations can be written in matrix form. The components of the precession matrix may be expressed as slowly-varying polynomials in time and we use the expressions given on page 34 of the Explanatory Supplement. We adopt a conventional matrix notation.

$$\pi_{11} = X_x \qquad \pi_{21} = X_y \qquad \pi_{31} = X_z$$

[6] $$\pi_{12} = Y_x \qquad \pi_{22} = Y_y \qquad \pi_{32} = Y_z$$

$$\pi_{13} = Z_x \qquad \pi_{23} = Z_y \qquad \pi_{33} = Z_z$$

The components of the nutation matrix may be written in the form given on page 43 of the Explanatory Supplement or in the rigorous form given below.

$$\nu_{11} = + \cos\Delta\psi$$
$$\nu_{12} = - \cos\varepsilon_o.\sin\Delta\psi$$
$$\nu_{13} = - \sin\varepsilon_o.\sin\Delta\psi$$

$$\nu_{21} = + \cos\varepsilon.\sin\Delta\psi$$
$$\nu_{22} = + \cos\varepsilon.\cos\varepsilon_o.\cos\Delta\psi + \sin\varepsilon.\sin\varepsilon_o$$
$$\nu_{23} = + \cos\varepsilon.\sin\varepsilon_o.\cos\Delta\psi + \sin\varepsilon.\cos\varepsilon_o$$

$$\nu_{31} = + \sin\varepsilon.\sin\Delta\psi$$
$$\nu_{32} = + \sin\varepsilon.\cos\varepsilon_o.\cos\Delta\psi + \cos\varepsilon.\sin\varepsilon_o$$
$$\nu_{33} = + \sin\varepsilon.\sin\varepsilon_o.\cos\Delta\psi + \cos\varepsilon.\cos\varepsilon_o$$

where $\Delta\psi$ and $\Delta\varepsilon$ are respectively the nutation in longitude and in obliquity and are evaluated using the series given on pages 44-45 of the Explanatory Supplement.

ε_o = mean obliquity of date

$\varepsilon = \varepsilon_o + \Delta\varepsilon$ = true obliquity of date

We may combine the precession matrix and the nutation matrix for any given date to obtain the precession-nutation matrix Π defined by

$$\Pi_{ij} = \nu_{i1}\pi_{1j} + \nu_{i2}\pi_{2j} + \nu_{i3}\pi_{3j}. \qquad [7]$$

4.2.2 TRANSLATION OF THE ORIGIN TO THE OBSERVER

This is a very simple transformation : to obtain the topocentric coordinates of the satellite we add the Saturnicentric coordinates of that satellite to the topocentric coordinates of Saturn, which have been determined as part of the light time calculation. At this stage, all coordinates should be referred to the True Equator and Equinox of Date.

4.2.3 ABERRATION

The effect called aberration arises from the fact that the observer has a small but finite velocity relative to the observed object. Thus the observed direction to the object is not the same as its instantaneous geometrical direction but is given by the resultant of the velocity vector of the light from the object and the velocity of the observer relative to it.

A detailed account of aberration may be found in Smart, Brouwer and Clemence, Explanatory Supplement and other works on positional astronomy. It is sufficient to note here that aberration can change the position of objects near the ecliptic by up to 20 arc seconds. However, its effect varies only a little over a small area of the sky and so we need only consider the effect of differential aberration. A table for calculating differential aberration is given on pages 52-53 of the Explanatory Supplement. For objects near the ecliptic the following maximum values apply (Astronomical Almanac page B21).

$$\text{Change in } \Delta\alpha/0''.01 = -0.570\ \cos(H+\alpha)\sec\delta.\Delta\alpha \\ -0.570\ \sin(H+\alpha)\sec\delta\tan\delta.\Delta\delta$$

$$\text{Change in } \Delta\delta/0''.01 = +0.570\ \sin(H+\alpha)\sin\delta.\Delta\alpha \\ -0.570\ \cos(H+\alpha)\cos\delta.\Delta\delta$$

where $\Delta\alpha$ and $\Delta\delta$ are in arc-minutes.

(Note that the coefficient 0.570 becomes 0.00950 when $\Delta\alpha$ and $\Delta\delta$ are in arc-seconds.)

For $\delta = +23^\circ$ we have

$$\Delta\alpha/0''.01 = -0.62\ \cos(H+\alpha).\Delta\alpha \\ -0.26\ \sin(H+\alpha).\Delta\delta$$

$$\Delta\delta/0''.01 \quad = +0.22 \sin(H+\alpha).\Delta\alpha$$
$$-0.52 \cos(H+\alpha).\Delta\delta.$$

For Iapetus the separation can be approximately 9'.5 giving $\Delta\alpha \simeq$ 10'.3/1.414 and $\Delta\delta \simeq$ 9'.5/1.414 so that the maximum effect is of the order of 0".06 for the outermost satellite .

We can regard the effect of differential aberration as proportional to the separation and so it is most important for the outer satellites. A change of 0".06 in the relative positions of two satellites should not be neglected in the reduction of the observations.

We calculate the effect of aberration upon the positions of the satellites using the formulae adapted from pages 156-160 of the Explanatory Supplement.

Writing $\Delta\alpha$ = correction to the Right Ascension and $\Delta\delta$ = correction to the Declination then

$$\Delta\alpha = h.\sin(H+\alpha_o).\sec\delta_o$$
[8]
$$\Delta\delta = h.\cos(H+\alpha_o).\sin\delta_o + i.\cos\delta_o$$

where $h.\sin H = C$

$h.\cos H = D$

$i = C.\tan\varepsilon = 0.43382.C$ for practical purposes.

C and D are the aberration Day Numbers formed from the velocity of the Earth <u>relative</u> to Saturn. The heliocentric velocity of the Earth is calculated using the subroutine BARVEL (Stumpff 1980) whilst the velocity of Saturn is calculated by a 9-point Lagrange differentiation formula operating upon a set of heliocentric coordinates tabulated at equal intervals. Denoting by (x', y', z') the components of the velocity of the Earth relative to Saturn, referred to the Mean Equator and Equinox of Date, then the Day Numbers are given by :

[9] $$C = +y'/c$$ $$D = -x'/c$$

where $1/c = 5.7756.10^{-3}$ days/AU.

As in the calculation of the Day Numbers in the Astronomical Almanac, the motions of the Earth and of Saturn are assumed to lie entirely within the plane of the ecliptic. Saturn's orbit is inclined at about $2^\circ.5$ to the ecliptic and this introduces an error of approximately $(V/c)\tan 2^\circ.5$ into the aberration in the Declination, where V is the orbital speed of Saturn. The relative size of this error compared to the total effect of aberration is $(V/U)\tan 2^\circ.5$ where U is the orbital speed of the Earth. This amounts to 0.014 or about 1% of the total aberration. The greatest error that this will produce in the differential aberration for any satellite pair is thus $0''.06 \times 0.014 = 0''.0008$ which is entirely negligible.

4.2.4 REFRACTION

The variation of the refractive index of the Earth's atmosphere causes the light rays from astronomical objects to be deflected during the passage through the atmosphere. A thorough account of refraction can be found in Smart (1978) and we adopt the formula given by Smart, namely

[10] $$R = R(\zeta) = z - \zeta = 58''.294 \tan\zeta \ - \ 0''.0668 \tan^3\zeta$$

where z = true zenith distance (i.e. unaffected by refraction)

ζ = observed zenith distance.

Thus the effect of refraction is to increase the zenith distance of an object by an amount R. This causes the observed Right Ascension and Declination of the object to be changed by a small amount $\Delta\alpha$, $\Delta\delta$.

As in the case of aberration, we are concerned with differential effects and thus with the change in refraction over a small area of the sky. Consider two objects close together but with different zenith distances. Using the notation above we may write

$$z_1 = \zeta_1 + R(\zeta_1)$$
$$z_2 = \zeta_2 + R(\zeta_2).$$

Thus

[11] $$z_2 - z_1 = \zeta_2 - \zeta_1 + R(\zeta_2) - R(\zeta_1)$$

which we may write to first order as

$$[12] \qquad z_2 - z_1 = \zeta_2 - \zeta_1 + (\zeta_2 - \zeta_1) \frac{dR}{d\zeta}.$$

$$= (\zeta_2 - \zeta_1)(1 + \frac{dR}{d\zeta})$$

where $dR/d\zeta$ is evaluated at $\zeta = \zeta_1$.

Writing $R = A.\tan\zeta + B.\tan^3\zeta$ then we have

$$[13] \qquad \frac{dR}{d\zeta} = A + (A + 3B)\tan^2\zeta + 3B\tan^4\zeta$$

and the coefficients A and B are, in radians, $2.8262.10^{-4}$ and $-3.24.10^{-7}$ respectively.

The difference in the zenith distance between the two objects is affected by refraction in the following way.

$$(z_2 - z_1) - (\zeta_2 - \zeta_1)$$

$$= \qquad (\zeta_2 - \zeta_1)(A + (A + 3B)\tan^2\zeta + 3B\tan^4\zeta)$$

To second order we have

$$[14] \qquad z_2 - z_1 = (\zeta_2 - \zeta_1)(1 + \frac{dR}{d\zeta}) + \frac{1}{2}(\zeta_2 - \zeta_1)^2 \frac{d^2R}{d\zeta^2}$$

and

$$[15] \qquad \frac{d^2R}{d\zeta^2} = 2(A + 3B)\tan\zeta + (2A + 18B)\tan^3\zeta + 12B\tan^5\zeta.$$

As an example of the magnitude of the effect of differential refraction we may take the extreme case of two objects only 30° above the horizon and differing in elevation by 600 arc seconds.

Thus $\zeta_1 = 60°$ and $\zeta_2 - \zeta_1 = 2.9\times10^{-3}$ radians

$$dR/d\zeta = 1.12\times10^{-3}$$

$$d^2R/d\zeta^2 = 3.82\times10^{-3}.$$

Hence the first-order term is 3.2×10^{-6} radians = 0".67 while the second-order term is 1.6×10^{-8} = 0".003. The first-order effect of refraction is to decrease the difference in zenith distance by about one part in a thousand. Clearly, the second-order term is quite negligible.

In practise we wish to calculate the effect of refraction upon the observed RA and Dec of the objects, and hence upon their relative positions.

We begin by writing the formulae given in the Explanatory Supplement (page 26) which relate the elevation and azimuth of an object to its Declination and Local Hour Angle (h = Local Sidereal Time - R.A.).

$$\begin{aligned} \cos\delta \sin h &= -\cos a \cos A &&= \ell \\ \cos\delta \cos h &= \sin a \cos\phi - \cos a \cos A \sin\phi &&= m \qquad [19] \\ \sin\delta &= \sin a \sin\phi + \cos a \cos A \cos\phi &&= n \end{aligned}$$

$$\cos a \sin A = -\cos\delta \sin h$$

[20] $$\cos a \cos A = \sin\delta \cos\phi - \cos\delta \cos h \sin\phi$$

$$\sin a = \sin\delta \sin\phi + \cos\delta \cos h \cos\phi$$

We take the partial derivatives of the first two equations with respect to the elevation a

$$-\sin\delta \sin h \frac{\partial\delta}{\partial a} + \cos\delta \cos h \frac{\partial h}{\partial a} = \frac{\partial \ell}{\partial a}$$

[21]

$$-\sin\delta \cos h \frac{\partial\delta}{\partial a} - \cos\delta \sin h \frac{\partial h}{\partial a} = \frac{\partial m}{\partial a}$$

which may be solved to yield

$$\sin\delta \frac{\partial\delta}{\partial a} = -\sin h \frac{\partial \ell}{\partial a} - \cos h \frac{\partial m}{\partial a}$$

$$\cos\delta \frac{\partial h}{\partial a} = -\sin h \frac{\partial m}{\partial a} + \cos h \frac{\partial \ell}{\partial a}.$$

It is preferable to employ $\partial n/\partial a$ in order to evaluate $\partial\delta/\partial a$. Thus

[22] $$\cos\delta \frac{\partial\delta}{\partial a} = \frac{\partial n}{\partial a}$$

because the factor sin δ causes problems for objects near to the celestial equator.

We also re-write $\partial h/\partial a$ as $-\partial\alpha/\partial a$ and we finally obtain

$$\cos^2\delta \ \frac{\partial\alpha}{\partial a} = -\cos\phi \sin A$$

[20]

$$\cos\delta \ \frac{\partial\delta}{\partial a} = \sin\phi \cos a - \cos\phi \sin a \cos A.$$

Using these derivatives we may calculate the change in α and δ due to a small change (Δa) in the elevation :

$$\Delta\alpha = \frac{\partial\alpha}{\partial a} \Delta a = \frac{\partial\alpha}{\partial a} \times R$$

[21]

$$\Delta\delta = \frac{\partial\delta}{\partial a} \Delta a = \frac{\partial\delta}{\partial a} \times R.$$

These formulae represent first-order corrections. Rigorous corrections may be obtained by calculating the pre-refraction elevation and azimuth of each object, adding the refraction to the elevation and then re-calculating the hour-angle and declination. However, first-order corrections are adequate for all practical purposes.

4.2.5 POSITION ANGLE AND SEPARATION

Most of the observations of the satellites of Saturn made in the period 1874 to 1947 are in the form of position angle (P) and separation (s) of one satellite relative to another. We must relate such P and s measures to the topocentric RA and Dec of a pair of satellites.

Consider the topocentric spherical triangle whose vertices are defined by the two satellites (denoted A and B) and the North Celestial Pole. We require the position angle of B, the observed satellite, with respect to A, the reference satellite.

In Figure 10 on page 107 the arc PA is the co-declination of A, that is $90° - \delta_A$, and PB is the co-declination of B.

The angle APB is the difference in the Right Ascensions of the two satellites : $\alpha_B - \alpha_A$. Note that the order of the satellites is important in this angle. The difference must be formed in the sense observed minus reference.

The arc AB is the separation of the two satellites and the angle PAB is the position angle of B with respect to A. Position angle is measured from North towards East, that is in an anti-clockwise sense as indicated in the diagram.

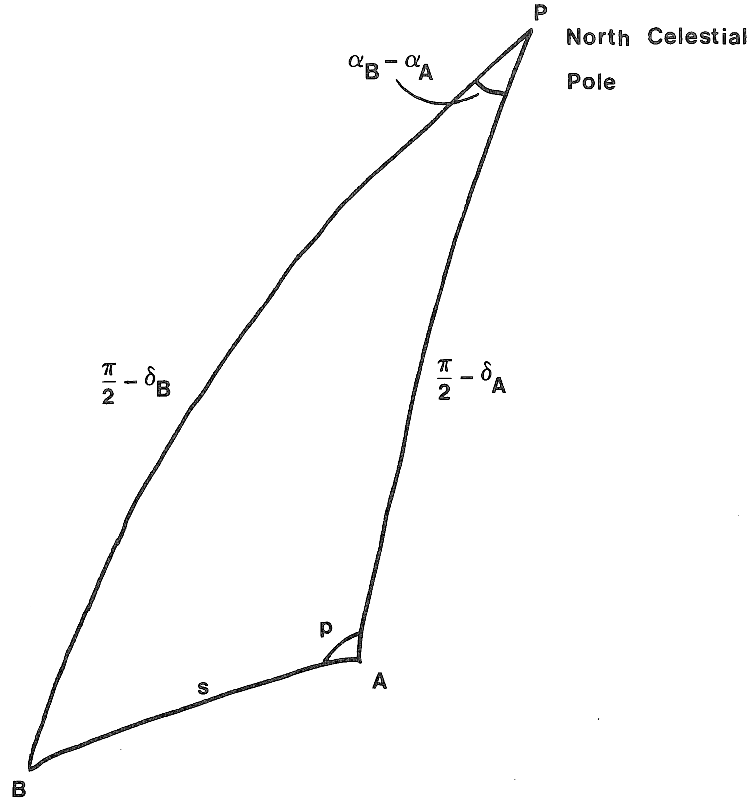


Figure 10. Position Angle and Separation

Using the formulae of spherical trigonometry (Explanatory Supplement page 472) we may write

$$\begin{aligned} \lambda &= \sin s \sin p &&= \cos\delta_B \sin(\alpha_B - \alpha_A) \\ \mu &= \sin s \cos p &&= \sin\delta_B \cos\delta_A - \cos\delta_B \sin\delta_A \cos(\alpha_B - \alpha_A) \\ \nu &= \cos s &&= \sin\delta_B \sin\delta_A + \cos\delta_B \cos\delta_A \cos(\alpha_B - \alpha_A). \end{aligned} \qquad [22]$$

Since sin s is always positive, we may deduce the quadrant of p by taking the sign of λ as the sign of sin p and the sign of μ as the sign of cos p. The following table gives the appropriate quadrant for p.

λ	μ	p	Quadrant
+	+	$0° \le p \le 90°$	1^{st}
+	-	$90° \le p \le 180°$	2^{nd}
-	-	$180° \le p \le 270°$	3^{rd}
-	+	$270° \le p \le 360°$	4^{th}

4.2.6 FIRST-ORDER CORRECTION TO P AND S FOR ABERRATION AND REFRACTION

If we neglect the effects of aberration and refraction upon the topocentric Right Ascension and Declination of the satellites then the deduced position angle and separation will be in error by a small amount, typically up to 0".7 for the outermost satellite Iapetus. Analysis of the available data has shown that the correction to be made to p and s for the combined effects of aberration and refraction generally does not exceed 0.2% of the datum itself even for objects observed at elevations less than 40°. For this reason we may regard aberration and refraction as small corrections to position angle and separation due to small changes in the R.A. and Dec of the satellites. If we suppose that aberration and refraction combine to alter the topocentric R.A. and Dec of the satellites by amounts $\Delta\alpha_A$, $\Delta\delta_A$, $\Delta\alpha_B$, $\Delta\delta_B$ then we may write

$$\Delta p = \frac{\partial p}{\partial \alpha_A} \Delta\alpha_A + \frac{\partial p}{\partial \delta_A} \Delta\delta_A + \frac{\partial p}{\partial \alpha_B} \Delta\alpha_B + \frac{\partial p}{\partial \delta_B} \Delta\delta_B$$

[23]

$$\Delta s = \frac{\partial s}{\partial \alpha_A} \Delta\alpha_A + \frac{\partial s}{\partial \delta_A} \Delta\delta_A + \frac{\partial s}{\partial \alpha_B} \Delta\alpha_B + \frac{\partial s}{\partial \delta_B} \Delta\delta_B .$$

In order to determine Δp and Δs it is evident that we need the partial derivatives of p and s with respect to the R.A. and Dec of the two satellites. These derivatives are also required in the differential correction process which is used to calculate improved values of the parameters of the dynamical model, and expressions for these partial derivatives are given in the following sections.

4.3 COMPARISON OF OBSERVATION WITH THEORY : DIFFERENTIAL CORRECTION

Comparison of a dynamical theory with observational data serves two purposes.

1. It allows us to evaluate how well the theory represents the dynamics of the real system which it is intended to model.

2. It enables us to improve the theory so that it is a closer model of the real system. This improvement may take the form of additional terms in an analytic theory, but more often it involves making small corrections to the numerical values of the parameters upon which the theory is based (i.e. orbital elements, or starting conditions of a

numerical integration). This method of improving a dynamical theory can be put into practise using the technique of differential corrections to the parameters.

Suppose we have a theory which provides the positions of a number of satellites at any given time t. The theory contains a set of parameters (orbital elements) $e_1, e_2, \ldots, e_N$ and so in a mathematical sense the coordinates of each satellite are functions of those parameters and also of the time argument. Any observable quantity, say a position angle p, is a function of the coordinates of the satellites and hence it is also a function of the parameters of the theory via the coordinates.

With a chosen set of parameters we may calculate the value of the observed quantity at some instant using the dynamical theory. This is called the computed ('C') value of p and it is denoted as p_c. At the same instant we have a value of p which has been obtained by observing the real satellite system. This is the observed ('O') value and is denoted as p_o. The difference between the two, in the sense observed-minus-computed, is called the O-C or residual and is denoted as Δp :

[24] $$\Delta p = p_o - p_c$$

We assume that this difference is due mainly to errors in the adopted values of the parameters and we seek to make corrections to the parameters $\Delta e_1, \Delta e_2, \ldots, \Delta e_N$ where

[25] $$\Delta e_i = (e_i)_o - (e_i)_c$$

is the difference between the 'correct' or 'best' value and the value adopted in the theory. Thus we may write

$$[26] \qquad \Delta p = \frac{\partial p}{\partial e_1} \Delta e_1 + \frac{\partial p}{\partial e_2} \Delta e_2 + \ldots + \frac{\partial p}{\partial e_N} \Delta e_N \quad .$$

In this equation, the left-hand side is a known quantity, as are the derivatives $\partial p/\partial e_1$, $\partial p/\partial e_2$ etc. The corrections Δe_1, Δe_2, ..., Δe_N are not known : they are the quantities which we wish to determine.

Such an equation is known as an equation of condition and it is the basis of differential correction theory. Each observation yields an equation of condition. In this context, a position angle measurement and a separation measurement are regarded as separate observations and each can be used to form an equation of condition.

When we have a large number of observations, we can form many equations of condition and so we have a number of simultaneous linear equations whose unknowns are the correction Δe_i. In most cases, we choose to combine the equations of condition into a set of normal equations in order to obtain a least-squares solution.

In the following sections we derive expressions for the partial derivatives of observed quantities with respect to the Saturnicentric coordinates of the satellites produced by the dynamical models. Such derivatives may be used with any theory which gives rectangular coordinates of the satellites in a fixed Saturnicentric reference frame.

4.4 PARTIAL DERIVATIVES OF POSITION ANGLE AND SEPARATION

In this section we derive the expressions for the partial derivatives of position angle and separation with respect to the Saturnicentric coordinates of the two observed objects. These coordinates are referred to the True Equator and Equinox of Date, but both they and the partial derivatives may be readily be converted to the fixed reference frame of the theory or the integration.

Consider the spherical triangle (on the topocentric sky sphere) formed by the North Celestial Pole and the two satellites. Refer to the diagram, which shows the triangle as seen by the observer from the inside of the sphere. Let the topocentric spherical coordinates of the objects be ρ_A, α_A, δ_A and ρ_B, α_B, δ_B and let p,s be the position angle and separation of B relative to A. Position angle on the celestial sphere is measured from the north, eastwards. Then we have

$$\begin{aligned} \lambda &= \sin s \sin p &&= \cos\delta_B \sin(\alpha_B - \alpha_A) \\ \mu &= \sin s \cos p &&= \sin\delta_B \cos\delta_A - \cos\delta_B \sin\delta_A \cos(\alpha_B - \alpha_A) \\ \nu &= \cos s &&= \sin\delta_B \sin\delta_A + \cos\delta_B \cos\delta_A \cos(\alpha_B - \alpha_A). \end{aligned} \qquad [27]$$

By differentiating λ and μ with respect to any parameter ω and rearranging, we obtain

$$\sin s \frac{\partial p}{\partial \omega} = \cos p \frac{\partial \lambda}{\partial \omega} - \sin p \frac{\partial \mu}{\partial \omega}$$

[28]

$$\cos s \frac{\partial s}{\partial \omega} = \sin p \frac{\partial \lambda}{\partial \omega} + \cos p \frac{\partial \mu}{\partial \omega}$$

and it is evident that

$$\frac{\partial p}{\partial \lambda} = \frac{\cos p}{\sin s} \qquad \frac{\partial p}{\partial \mu} = - \frac{\sin p}{\sin s}$$

[29]

$$\frac{\partial s}{\partial \lambda} = \frac{\sin p}{\cos s} \qquad \frac{\partial s}{\partial \mu} = \frac{\cos p}{\cos s}.$$

We form the derivatives of λ and μ with respect to the R.A. and Dec of the two objects thus :

$$\frac{\partial \lambda}{\partial \alpha_A} = - \frac{\partial \lambda}{\partial \alpha_B} = - \cos \delta_B \cos (\alpha_B - \alpha_A)$$

[30]

$$\frac{\partial \lambda}{\partial \delta_A} = 0$$

$$\frac{\partial \lambda}{\partial \delta_B} = - \sin \delta_B \sin (\alpha_B - \alpha_A)$$

$$\frac{\partial \mu}{\partial \alpha_A} = - \frac{\partial \mu}{\partial \alpha_B} = - \lambda \sin \delta_A$$

[31]

$$\frac{\partial \mu}{\partial \delta_A} = - \nu$$

$$\frac{\partial \mu}{\partial \delta_B} = \cos \delta_B \cos \delta_A + \sin \delta_B \sin \delta_A \cos (\alpha_B - \alpha_A)$$

and we may now calculate the derivatives of p and s with respect to α_A, δ_A, α_B, δ_B.

$$\frac{\partial p}{\partial \alpha_A} = \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha_A} + \frac{\partial p}{\partial \lambda}\frac{\partial \mu}{\partial \alpha_A} \qquad [35]$$

and likewise for the other derivatives.

We may seek the derivatives of the R.A. and Dec with respect to the Saturnicentric coordinates of the satellites. For each satellite we have

$$\begin{aligned} \rho \cos\delta \cos\alpha &= x + X \\ \rho \cos\delta \sin\alpha &= y + Y \qquad [36] \\ \rho \sin\delta &= z + Z \end{aligned}$$

where (x,y,z) are the Saturnicentric coordinates of the satellite and (X,Y,Z) are the topocentric coordinates of Saturn, all referred to the True Equator and Equinox of Date.

We obtain the following derivatives :

$$\begin{aligned} \frac{\partial \alpha}{\partial x} &= -\frac{\sin\alpha}{\rho\cos\delta} \\ \frac{\partial \alpha}{\partial y} &= +\frac{\cos\alpha}{\rho\cos\delta} \qquad [37] \\ \frac{\partial \alpha}{\partial z} &= 0 \end{aligned}$$

$$\frac{\partial\delta}{\partial x} = -\frac{\sin\delta\cos\alpha}{\rho}$$

[38] $$\frac{\partial\delta}{\partial y} = -\frac{\sin\delta\sin\alpha}{\rho}$$

$$\frac{\partial\delta}{\partial z} = \frac{\cos\delta}{\rho}$$

We may form these derivatives for both satellites and then we combine them with derivatives such as $\partial p/\partial\alpha$ to obtain

[39] $$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial\alpha}\frac{\partial\alpha}{\partial x} + \frac{\partial p}{\partial\delta}\frac{\partial\delta}{\partial x}$$

and so forth. Note that α_A and δ_A are independent of x_B, y_B, z_B, and α_B and δ_B are independent of x_A, y_A, z_A. Hence we do not include terms such as $\partial\alpha_A/\partial x_B$ since they are zero.

At this point we have the partial derivatives of position angle and separation with respect to the Cartesian coordinates of the satellites in a system where all quantities are referred to the True Equator and Equinox of Date. The dynamical model of the satellite system is referred to a fixed coordinate system and so we must transform the derivatives to that coordinate system. We recall that the coordinates in the reference frame of the dynamical theory may be related to the True Equator and Equinox of Date via a transformation matrix which incorporates the instantaneous effect of precession and nutation and (in the case of the numeric integration) a constant rotation transformation. If we denote this matrix by M then we may write

$$[37] \quad \begin{aligned} x &= M_{11}X + M_{12}Y + M_{13}Z \\ y &= M_{21}X + M_{22}Y + M_{23}Z \\ z &= M_{31}X + M_{32}Y + M_{33}Z \end{aligned}$$

where (x,y,z) are the components of a vector referred to the True Equator and Equinox of Date and (X,Y,Z) are the components of that vector in the reference frame of the dynamical model.

Now for any observed quantity p, we have

$$[38] \quad \frac{\partial p}{\partial X} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial X}$$

and likewise for $\partial p/\partial Y$ and $\partial p/\partial Z$.

The derivatives $\partial x/\partial X$, $\partial y/\partial X$, $\partial z/\partial X$ etc. may be recognised as the components of the matrix M. For example :

$$[39] \quad \begin{aligned} \partial x/\partial X &= M_{11} \\ \partial y/\partial X &= M_{21} \\ \partial z/\partial X &= M_{31}. \end{aligned}$$

We may now calculate the partial derivatives of position angle and separation with respect to the Saturnicentric coordinates produced by the dynamical model. These are the derivatives that were sought in this section.

5.0 NUMERICAL INTEGRATION

5.1 INTRODUCTION

It was decided to model the dynamics of the three outer satellites of Saturn (Titan, Hyperion and Iapetus) using a numerical integration. This method has been employed by Sinclair and Taylor who have fitted an integration to astrometric observations made during the period 1967 to 1983.

In this chapter we describe the fitting of a numerical integration to visual observations of the three satellites over the period 1874 to 1947. Such visual observations have not been analysed using numerical integration before and this work represents a significant development in the study of natural satellite dynamics. Apart from being a valuable exercise in its own right, it is an important preliminary to the goal of linking pre-1947 visual observations with post-1967 photographic observations in a global solution involving a numerical integration of the satellite orbits over a period of 120 years.

The reasons for adopting numerical integration as a dynamical model are as follows :

(1) As long as we include all significant gravitational effects in the force model then a numerical integration provides a dynamically consistent representation of a satellite system. In this sense it has an advantage over analytic theories whose series must be truncated after a relatively small number of terms if they are to be convenient to use and easy to develop.

(2) The number of free parameters in a numerical integration can be restricted to the dynamically consistent minimum set : a position vector and velocity vector for each satellite plus the masses of the disturbing bodies and the form factors of the primary. In this way, we may avoid the introduction of pseudo-arbitrary parameters which are adopted in some analytic theories and which allow the least-squares fitting process to give artificially small root-mean-square residuals.

(3) The analytic theory of Hyperion is a problem of great complexity. Newcomb has placed it second only to the lunar theory in terms of the difficulty of its formulation. It has been studied by Newcomb, Woltjer and Message and is currently the subject of work by Message, Taylor and Sinclair (see for example Taylor (1984)).

Numerical integration of the motion of Hyperion, by contrast, presents no more difficulty than for any other satellite and hence allows observations of Hyperion to be analysed together with those of Titan and

Iapetus. The results of such an analysis will not be prejudiced by the shortcomings of an analytic theory.

(4) Once we have found a set of parameters with which the integration gives the closest fit to the observations, we may regard an integration based upon these parameters as effectively representing the observations. The information which was contained in the observations is now inherently contained in the parameters <u>in conjunction</u> with the chosen integration algorithm.

This means that we can use an ephemeris produced by the integration as a model to improve the analytic theories of the satellites. Comparison of the theories with the numerical integration may give clues which point to deficiencies and omissions from the theories. This approach has been used (Harper et al.) to identify Solar terms in the theory of Iapetus.

5.2 THE NUMERICAL INTEGRATION PROGRAM 'TITAN'

The numerical integration method used to generate ephemerides of the outer satellites of Saturn is an 8th order central-difference Gauss-Jackson scheme. It employs an iterative starting procedure. A predictor cycle followed by a corrector cycle is used in the integration of the equations for the coordinates of the satellites. In the integration of the equations for the partial derivatives of the coordinates with respect to the parameters, a predictor cycle alone is used.

The program which carries out the integration was written by Dr.A.T. Sinclair at the Royal Greenwich Observatory (see acknowledgements) for use on the Observatory's VAX 11-750 computer. It has been modified to enable it to run on the IBM 3083 of the University of Liverpool Computer Laboratory.

5.2.1 THE COORDINATE SYSTEM OF THE NUMERICAL INTEGRATION

We integrate the rectangular coordinates of the satellites in a Saturnicentric coordinate system whose xy-plane is the equator plane of Saturn. This is assumed to be a fixed plane over the time-span of the integration. The x-axis of the system is in the direction of the ascending node of the equator of Saturn upon the Earth's mean equator and equinox of 1950. The transformation between the integration frame and the mean equator and equinox of 1950 may thus be represented by a constant matrix as described in a previous chapter.

The choice of the equator plane of Saturn as the xy-plane of the integration arises from two considerations.

- The calculation of the perturbations due to the oblateness of Saturn are simplified because of symmetry about the xy-plane.

- There are no other similarly preferred planes in the satellite system and so any reference system is equally convenient from the computational point of view.

5.2.2 THE FORCE MODEL

The accelerations acting upon each satellite in the numerical integration are as follows :

(1) The gravitational attraction of Saturn regarded as a point mass.

This obeys a simple inverse square law which may be written as

[1] $$\underline{a}_i^0 = - GM \frac{(1 + m_i)}{r_i^3} \underline{r}_i$$

where $\underline{r}_i$ = Saturnicentric position vector of the i^{th} satellite

G = gravitational constant

M = mass of Saturn

m_i = mass ratio of the i^{th} satellite to Saturn

$r_i = |\underline{r}_i|$.

(2) The gravitational attraction of the other satellites.

The acceleration upon the i^{th} satellite due to the attraction of the j^{th} satellite may be written

$$[2] \qquad \underline{a}_{ij} = G\,M\,m_j \left\{ \frac{\underline{r}_j - \underline{r}_i}{r_{ij}^{\,3}} - \frac{\underline{r}_j}{r_j^{\,3}} \right\}$$

where
m_j = mass ratio of the j^{th} satellite to Saturn
$\underline{r}_j$ = Saturnicentric position vector of the j^{th} satellite
r_{ij} = the distance between the i^{th} and j^{th} satellites.

The first term is the gravitational attraction itself ; the second term is called the 'indirect term' and arises from the fact that the origin of the coordinate system is the centre of mass of Saturn and not the barycentre of the entire system. The second term is the acceleration of the centre of mass of Saturn relative to the barycentre due to the attraction of the j^{th} satellite upon Saturn. This expression is derived in appendix C.

Titan is the most massive satellite in the system and it perturbs Hyperion and Iapetus quite strongly. Indeed, the motion of Hyperion is characterised by the perturbation by Titan. Of the other satellites, Iapetus is the more massive and its perturbations on the motion of Titan and Hyperion are included in the force model. Perturbations by Rhea are also included. The most significant effect of Rhea is upon the secular rates of the nodes and apsides of the outer satellites, especially Titan. In this respect, it augments the perturbations due to the oblateness of Saturn. However, to allow for periodic perturbations by Rhea in the motion of Titan, the orbit of Rhea is assumed to be circular and fixed in the equator plane of Saturn. In the reference frame of the integration, the position of Rhea at time t is given by

$$X_R = 0.0035232 \cos L$$
$$Y_R = 0.0035232 \sin L$$
$$Z_R = 0$$

where $L = 231^\circ.761 + 79^\circ.69004007\ (t - 2411093.0)$

$$\mu_R = 4.4\ 10^{-6}.$$

The elements of Rhea are from Sinclair (1977) and μ_R, the mass ratio Rhea : Saturn is the value used by Sinclair and Taylor (1985) and taken from Tyler et al (1981).

Perturbations by Hyperion upon Titan were not included in the force model as Hyperion is not massive enough to affect the motion of either of the other satellites.

(3) The gravitational attraction of the Sun.

We may regard the sun as a very distant and very massive 'satellite' of Saturn. The acceleration of the i^{th} satellite due to the attraction of the Sun is given by

[3] $$\underline{a}_{is} = G\,M\,m_s \left\{ \frac{\underline{r}_s - \underline{r}_i}{r_{is}^3} - \frac{\underline{r}_s}{r_s^3} \right\}$$

where μ_s = mass ratio Sun : Saturn

$\underline{r}_s$ = Saturnicentric position vector of the Sun

r_{is} = distance between the Sun and the i^{th} satellite.

As in the previous case, the second term is an indirect term representing the acceleration of the centre of mass of Saturn with respect to the barycentre of the Sun-Saturn-satellites system due to the gravitational attraction of the Sun. The derivation is given in appendix C.

The heliocentric coordinates of Saturn used in the calculation of the solar perturbations are derived from the ephemeris of Saturn in Astron. Pap. Amer. Eph. volume 12. The coordinates are transformed from the ephemeris reference frame to that of the integration and then turned into sets of Chebyshev coefficients covering successive 400-day intervals. The use of Chebyshev series to calculate the coordinates of the Sun relative to Saturn during the integration is intended to speed up the process of interpolation whilst maintaining accuracy in the interpolated coordinates.

(4) The oblateness of Saturn.

The disturbing function of Saturn due to its oblateness may be written as (cf. Herrick (1972), chapter 18)

[4] $$R_e = - GM/r \sum_{n=2}^{\infty} (a_e/r)^n J_n P_n(w)$$

where a_e = equatorial radius of Saturn

r = distance from the centre of Saturn

J_n = n^{th} zonal harmonic coefficient of the potential field

w = z/r = latitude of the point above the equatorial plane of Saturn

and $P_n(w)$ is the Legendre polynomial of degree n.

The acceleration upon a satellite at this point due to the oblateness of Saturn is

[5] $$\underline{a}_e = \nabla R_e$$

Since the coordinate system has been chosen so that Saturn is symmetric about the xy-plane, all odd-numbered harmonic coefficients (J_3, J_5 etc.) disappear. Moreover, the ratio $(a_e/r)^n$ and the harmonic coefficients J_n rapidly become smaller for large values of n and so in practise we may neglect harmonics beyond J_4. Thus R_e contains only contributions from the n=2 and n=4 terms in the expression given above. We may write

[6] $$R_e = -(GM/r) \times \{(a_e/r)^2 J_2 P_2(w) + (a_e/r)^4 J_4 P_4(w)\}.$$

The components of the acceleration upon the satellite are

$$[7] \qquad \ddot{x} = \partial R_e/\partial x, \quad \ddot{y} = \partial R_e/\partial y, \quad \ddot{z} = \partial R_e/\partial z$$

and we obtain the following expressions (cf. Sinclair and Taylor (1985)) for the acceleration. The identities used to simplify the Legendre polynomials are derived in appendix D.

$$[8] \qquad \underline{a}_e = A\,\underline{r} + B\hat{\underline{k}}$$

where $A = (GM/r^3)\,\{J_2\,(a_e/r)^2\,P'_3(w) + J_4\,(a_e/r)^4\,P'_5(w)\}$

$B = -(GM/r^2)\,\{J_2\,(a_e/r)^2\,P'_2(w) + J_4\,(a_e/r)^4\,P'_4(w)\}$

$\hat{\underline{k}}$ = the unit vector in the z direction

and $P'_n(w)$ denotes the first derivative of the Legendre polynomial with respect to its argument.

5.2.3 PARAMETERS OF THE NUMERICAL INTEGRATION MODEL

A number of parameters of the numerical integration are considered to be free parameters : their values may be altered from one integration run to another, usually as a result of a least-squares correction process involving comparison of the integration with data or with analytic theories of the motions of the satellites. These parameters are of two types.

1. Elements of satellite orbits. There are six arbitrary parameters in the motion of any satellite ; these may take the form of classical orbital elements, but in a numerical integration it is more convenient to use the position and velocity vectors of the satellites at some fixed epoch. For each satellite there are three position components and three velocity components. These are equivalent to six orbital elements ; indeed, it is a simple matter to convert classical elements to position and velocity vectors and vice versa (Herrick (1971) gives a particularly lucid account of the equivalence of the two different types of parameter set and the transformation between them).

   Since we are dealing with three satellites, their orbital elements provide us with eighteen free parameters.

2. Mass and form-factor parameters. In their comparison of astrometric observations of the outer satellites with a numerical integration, Sinclair and Taylor (1985) also treated the masses of Titan, Iapetus and Saturn as parameters to be determined by least-squares iterative fitting. In addition the dynamic form-factors of Saturn, J_2 and J_4 were included as free parameters.

   The set of free parameters now numbers 23.

- The position vector and velocity vector of Titan.

- The position vector and velocity vector of Hyperion.
- The position vector and velocity vector of Iapetus.
- The mass ratio Titan/Saturn.
- The mass ratio Iapetus/Saturn.
- The mass ratio Saturn/Sun.
- The dynamic form factors J_2, J_4.

These are the free parameters in the current work.

5.2.4 PARTIAL DERIVATIVES OF THE COORDINATES

In addition to the coordinates of the satellites, we require the partial derivatives of the coordinates with respect to each of the free parameters of the integration model. These partial derivatives are used in the determination of the parameters by fitting the integration to observations or to analytic theories.

The partial derivatives are calculated by numerical integration in the same way as the coordinates. From the force model we may deduce expressions for the second time derivatives of the partial derivatives, i.e.

$$\frac{d^2}{dt^2}\left(\frac{\partial X_i}{\partial q_k}\right)$$

where X_i is any of the 9 satellite coordinates

q_k is any of the 23 free parameters.

The integration scheme thus incorporates 9 coordinates and 9×23 = 207 partial derivatives. The partial derivatives are not required to the same accuracy as the coordinates and so to save computing time, they are integrated using a predictor step alone.

If we write

[9] $$F_i = d^2X_i/dt^2$$

that is, F_i is the acceleration of the coordinate X_i given by the force model, then we have

[10] $$\frac{d^2}{dt^2}\left(\frac{\partial X_i}{\partial q_k}\right) = \frac{\partial F_i}{\partial q_k} + \sum_{j=1}^{18} \frac{\partial F_i}{\partial X_j} \frac{\partial X_j}{\partial q_k}.$$

The first term on the right-hand side is the explicit derivative of F_i with respect to the parameter q_k. In the case where q_k is a component of an initial position or velocity vector, this will be zero since the acceleration is not an explicit function of the initial position and velocity components. However, there are explicit derivatives with respect to the free parameters which correspond to masses and form-factors since these parameters do appear directly in the force model.

The second term on the right-hand side represents the implicit dependence of F_i upon the parameter q_k via the coordinates. It contributes to all the partial derivatives.

The starting values of the partial derivatives in the integration are

[11] $$\frac{\partial X_i}{\partial q_i} = 1$$

where q_i is the initial value of the coordinate X_i and

[12] $$\frac{d}{dt}\left(\frac{\partial X_j}{\partial q_j}\right) = 1$$

where q_j is the initial value of the velocity component dX_j/dt.

All other starting values of derivatives are zero.

5.3 ITERATIVE FITTING METHOD

The parameters of the numerical integration model were determined by comparing visual observations of the three satellites with the predicted positions from the integration. The differences between the observed and computed positions were combined with the partial derivatives described in previous sections to produce an equation of condition for each observation which expresses the (small) corrections to be made to the parameters in terms of the observed-minus-computed residual. Repeated application of this method yields parameter sets which give successively better fits to the observations. This is the basis of iterative least-squares differential correction, the technique which is used in this work to fit the numerical integration to the observations. The stages in the process may be enumerated.

(1) We must obtain a starting set of parameters that will give a reasonably close fit to the observations. The differential correction process relies upon the first-order approximation

$$\Delta p = \frac{\partial p}{\partial e_1} \Delta e_1 + \frac{\partial p}{\partial e_2} \Delta e_2 + \ldots + \frac{\partial p}{\partial e_N} \Delta e_N$$

where p is the observed quantity and Δp is its observed-minus-computed residual, and Δe_1, Δe_2, ..., Δe_N are the small corrections to the parameters e_j.

The starting values must thus be obtained from a model which already matches the observational data quite closely. The analytic theories of the satellites are chosen for this purpose, specifically the theories which Sinclair and Taylor have fitted to astrometric observations covering the period 1967 to 1982. As an initial approximation, the Saturnicentric rectangular coordinates of each satellite are calculated from the analytic theories at intervals corresponding to 10° arcs for several dates around the chosen epoch of the integration. The velocity components are calculated using a 7-point Lagrange interpolation formula differentiated once with respect to the interpolation argument. The initial values of the mass parameters and the dynamic form factors of Saturn are taken from the work of Sinclair and Taylor (1985).

Numerical integration is then used to calculate the positions of the satellites at two thousand random dates in a twenty-year interval centred on the adopted zero epoch of the integration. The Saturnicentric rectangular coordinates of the satellites are compared with the coordinates given by the analytic theories for the same dates. Equations of condition are constructed for each of the nine coordinates ξ (three from each satellite) incorporating the difference

$$\Delta\xi = \xi_{\text{Analytic theory}} - \xi_{\text{Numerical integration}}$$

expressed in terms of the corrections to the parameters of the integration. Solution of the equations of condition yields an improved set of parameters for the integration. This process of fitting the integration to the analytic theories is repeated until convergence is obtained i.e.

the corrections to the parameters become very small. At this point, the positions of the satellites given by the numerical integration closely match those given by the analytic theories and are thus close to the real satellite system.

A set of random dates is used in this process in order to avoid a 'sampling effect' whereby significant short-period terms in the analytic theories are masked because their period matches the sampling period. A different set of random dates is generated for each iteration.

The mass parameters and form factors are not allowed to vary at this stage because solution for these parameters in addition to position and velocity components by comparison with the theories would give only the values implicit in the theories and these were not judged to be better than the initial values adopted for the integration.

We may now begin the task of fitting the integration to the observations.

(2) The Saturnicentric rectangular coordinates of the three satellites are calculated by numerical integration at intervals of 0.25 days for a period sufficient to cover all the observations to be used. In practise the integration is carried out for forty years either side of the zero epoch in two runs : one from 1910 to 1950 and the other from 1910 to 1870. The coordinates of the satellites at the moment of each observation are obtained by interpolation while the integration is in progress. A

fourth-order interpolation formula is employed, both for the coordinates and for the partial derivatives of the coordinates with respect to the parameters.

(3) The coordinates of the satellites are transformed from the reference frame of the integration, based upon the equator plane of Saturn, to the true equator and equinox at the date of the observation. The partial derivatives are also transformed. This transformation consists of two rotations : the first is from the reference frame of the integration to the mean equator and equinox of B1950 and is a fixed transformation ; the second is from the mean equator and equinox of B1950 to the true equator and equinox of date. This is different for each observation.

We now have the Saturnicentric coordinates of each satellite referred to the true equator and equinox of date. We add the topocentric position vector of the centre of Saturn to obtain the topocentric positions of the satellites. From these, we may deduce the Right Ascension and Declination of each satellite and of the centre of Saturn's disk, and hence the position angle and separation of any of the four objects with respect to any other. We calculate only the datum required by the observation, together with the partial derivatives of the datum with respect to the parameters of the integration. Thus we may construct an equation of condition for each observation by combining the partial derivatives with the residual, i.e. the difference <Observed datum> minus <Computed datum>.

(4) Each observation yields an equation of condition. These are collected and combined to give a set of normal equations according to the theory of least-squares. An equation of condition is added to the normal equations only if its O-C residual is less than a specified rejection limit. This rejection limit was initially set at a nominal figure of 2.0 arc-seconds and this was later adopted as the standard limit as the root-mean-square residuals converged to a value of approximately 0.6 arc-seconds. Thus the rejection limit was 3 times the RMS residual, corresponding to rejection of 0.3% of errors which are normally distributed.

The normal equations may now be solved to give the corrections to the parameters. As part of the solution process, the standard errors of the parameters and their cross-correlations may be determined. It is also possible to fix one or more of the parameters if it is evident that there are strong correlations. This was indeed found to occur : for example, the mass of Saturn was highly correlated with the semi-major axes of the satellite orbits and so it was decided to keep a fixed value for Saturn's mass.

It was also found that the correction to Saturn's J_4 form factor was several orders of magnitude larger than any reasonable value of the parameter itself, and so this was also kept constant.

(5) The O-C residuals are analysed according to satellite pair and datum type (position angle or separation). The root-mean-square, mean and standard deviation are calculated and a histogram drawn to show the dis-

tribution of residuals. This is a useful indication of the progress of the iterations : clearly, we want the residuals to show a Gaussian distribution about zero, with a very narrow peak. If the iteration scheme is not working then this will be shown by the histogram. Example histograms are given in a later section where the results are discussed.

When the corrections to the parameters are small compared to the standard errors then the iterative process is complete. Further iterations will not significantly improve the fit of the integration to the observations. However, when the corrections are larger than the standard errors, the process begins again at stage (2)

5.4 RESULTS. (1) PROBLEMS WITH NON-CONVERGENCE IN THE FIRST PHASE

In practise, the least-squares fitting process was not as straightforward as the previous section might suggest. There was some difficulty in obtaining convergence in successive iterations whilst fitting the numerical integration to the analytic theories during stage (1) of the iterative procedure described previously. We may denote the difference between the position of each satellite given by the numerical integration and that given by the analytic theory as

$$\xi = | \underline{r}_{Theory} - \underline{r}_{Integration} | .$$

At each iteration, a graph of the 2000 ξ values plotted against time for each satellite showed a clear systematic trend.

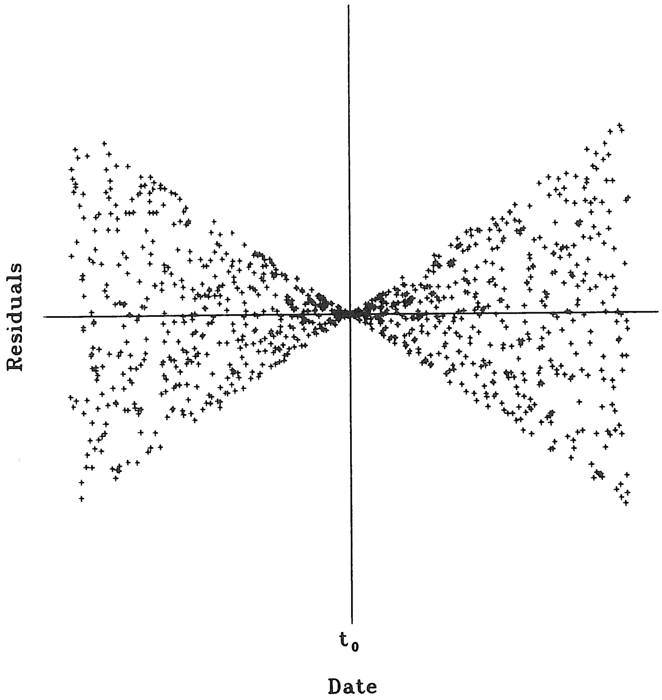


Figure 11. Integration-minus-theory residuals (schematic)

The maximum value of ξ increases linearly as a function of $|t - t_o|$, that is, in proportion to the distance from the initial epoch of the integration. This is the point at which we seek to determine starting values for the position and velocity of each satellite. The effect was most marked for Hyperion. When Hyperion was removed from the iterative process, convergence was achieved quite easily for Titan and Iapetus alone and satisfactory starting conditions were obtained for these satellites.

Since the analytic theory of Hyperion is such a complex problem, the short-period perturbations by Titan have been omitted from the computer program developed by Sinclair and Taylor to represent its motion ; these terms are generated separately by numerical integration and may be used as a look-up table in conjunction with the purely analytic part of the theory. However, these terms were not available for the time interval covered by the present study (1870 to 1947) and so the analytic theory of Hyperion used here is not complete. It is probable that this omission is the cause of the non-convergence ; we are trying to fit a dynamically consistent model, viz. the numerical integration, to one which is not consistent. The fitting process attempts to compensate by over-correcting the elements of Hyperion in the integration in successive iterations, and the result is divergence of the integration model from the analytic theory.

Titan is adversely affected by this because its parameters are linked to those of Hyperion by virtue of its large perturbations upon Hyperion. In mathematical terms, we may observe that equations of condition arising from the coordinates of Hyperion contain partial derivatives with respect to the parameters of Titan. Hence Titan's parameters may be subject to large and spurious corrections when Hyperion's position is allowed to diverge.

The solution adopted for this problem was to omit Hyperion from the iterative process of fitting the numerical integration to the analytic theories. Starting parameters were obtained for Titan and Iapetus by iteration, while those for Hyperion were calculated from the analytic theory without iteration. If these parameters were subsequently found to be inadequate then Hyperion could be omitted from comparison of the integration with observations.

Another possible cause for the divergence may be considered : the analytic theories employed as a reference model have been fitted to modern (post-1967) observations. The orbital elements, and in particular the longitude of the satellite at the epoch λ_o, are thus the best values for the period 1967 to 1983. In fitting the integration to these theories at the epoch 1910, we may be extrapolating the theories beyond their range of validity. This will be evident if the mean motions adopted in the theories are slightly in error. The mean motions used are values determined by Struve and Woltjer on the basis of pre-1935 observations covering a rather short time, 30 years at most. In the present work, the theories have been extended backwards some 70 years from the epoch of the data to which they have been fitted and so any error in the mean motion will be magnified as an error in the longitude of the satellite. The solution to this problem is to fit the theories to all the available data to obtain a better value of the mean motion, but this is a considerable exercise in its own right and it was not thought to be appropriate to undertake it in this context.

5.5 LEAST-SQUARES CORRECTION THEORY

The least-squares correction process is based upon the minimisation of the sum of the squares of the observed-minus-computed (O-C) residuals which are formed when the dynamical theory is compared with observations. Each O-C is the right hand side of an equation of condition

$$[13] \qquad \frac{\partial p^i}{\partial e_1} \varepsilon_1 + \quad \dots \quad + \frac{\partial p^i}{\partial e_N} \varepsilon_N = p_o^{\;i} - p_c^{\;i} = \Delta p^i$$

where $\varepsilon_j = \Delta e_j = (e_j)_{true} - (e_j)_{conjectured}$.

We wish, therefore, to minimise the function

$$[14] \qquad \rho = \quad (\Delta p^i)^2$$

with respect to each of the corrections $\varepsilon_1, \dots, \varepsilon_N$. This means that the derivatives of ρ with respect to each ε_j must be zero. That is,

$$[15] \qquad \partial p/\partial \varepsilon_j = 0 \qquad\qquad (j = 1, \dots, N).$$

This provides us with the set of N equations called the normal equations :

$$[16] \qquad a_{k1}\varepsilon_1 + \quad \dots \quad + a_{kN}\varepsilon_N = b_k$$

where

$$[17] \qquad a_{jk} = \sum_{i=1}^{N} \frac{\partial p^i}{\partial e_j} \frac{\partial p^i}{\partial e_k}$$

$$[18] \qquad b_j = \sum_{i=1}^{N} \frac{\partial p^i}{\partial e_j} \Delta p^i .$$

From these equations we note that the normal matrix (i.e. the matrix of the a_{jk}'s) is symmetric and positive definite. This implies that $a_{jk} = a_{kj}$ and $a_{jj} > 0$. The corrections ε_k may be determined by inverting the normal equations, which can be written in matrix notation as

$$[19] \qquad a\, \underline{\varepsilon} = \underline{b}.$$

Clearly,

$$[20] \qquad \underline{\varepsilon} = a^{-1}\underline{b} = s\, \underline{b}$$

where s is the inverse of the normal matrix.

5.5.1 STANDARD ERRORS

In a real physical system described by a parameterised model such as a numerical integration, there exists a set of parameters { $\underline{e}^{\,0}$} which best describes the system. This set is, as it were, the 'true' set of parameters and it is the set which would be determined if we possessed observations of the system which contained no errors.

In practise, of course, observations are prone to errors of several kinds. The most common (and most amenable to analysis) are random in nature and are due to any number of unforeseen and unpredictable effects on the part of the observer, his measuring apparatus and the conditions in which the observations are made. Such random errors are characterised by the fact that they form a Gaussian (normal) distribution when analysed in large numbers. Thus each observation consists of two parts : the 'true' value of the observed quantity in the sense used above, which we denote as y_i^0, and an error η_i taken from a normal distribution. The value observed in practise is y_i and we may write

[21] $$y_i = y_i^0 + \eta_i .$$

The crux of the problem facing us is this : we do not know the values of the η_i's.

The second type of error is not random. It is systematic in the sense that it affects the all observations in a similar way. It may be additive so that we have

[22] $$y_i = y_i^0 + \eta_i + \xi$$

where ξ is a constant, or it may be multiplicative so that

[23] $$y_i = \beta(y_i^0 + \eta_i)$$

where β is a constant, or it may be a combination of both types. In general, a systematic error has its cause in some feature of the experimental method which can be explained in physical terms without reference to random behaviour. As an example, consider a micrometer used to measure angular separation between satellites. If, whilst measuring separation between a satellite and a bright planet, the observer repeatedly overshoots the far limb of the planetary disk with the movable wire and does not realise the fact, then there will be a trend for the measures to be too large by a small amount whose value has a non-zero mean. Of course, careful observational technique is designed to minimise such effects but there may remain some systematic errors : a micrometer whose screw-thread has been incorrectly calibrated along part of its length may give rise to systematic multiplicative errors i.e. errors of scale. Moreover, such errors may be dependent upon the conditions under which the micrometer is used : ambient temperature changes causing different parts of the device to expand at different rates, to cite one example.

Systematic errors may manifest themselves in the O-C residuals. If this is the case, we can (and must) take steps to eliminate them by tracking down their physical cause.

It may, however, be more likely that the parameters of the model will adjust to incorporate these errors during the least-squares fitting process. Continuing the example of scale errors in separation measures : if the angular separations within a satellite system are systematically measured too large or too small by a given factor then a least-squares procedure based upon osculating orbital elements may respond by converging to a solution where the semi-major axes of the satellites are all too large or too small by a similar factor. If the mass of the primary is also a free parameter, it may also be in error by a corresponding factor.

We must, in the first instance, assume that any errors in the observations are random. After we determine the corrections to the parameters of the model using the least-squares method, we may then calculate the likely errors in the parameters due to errors in the observations upon which the solution is based. We quote the parameters in the form

$$e \pm \delta e$$

where e is the calculated value, and we are confident that the 'true' value lies in the interval $e-\delta e$ to $e+\delta e$ with a probability p. That is,

[24] $$e - \delta e \le e^0 \le e + \delta e$$

with probability p.

Following Brouwer and Clemence (1961, p.226) and Jeffreys (1939, p.61) we quote the standard error of each parameter, denoted by σ, which is the value of δe corresponding to $p = 0.683$. That is to say, it is 68.3% certain that the 'true' value of a parameter will lie within one standard error of its value determined from data whose errors are distributed in a random fashion following a normal distribution.

We may calculate the standard errors of the parameters as follows : after solving the normal equations to determine the corrections to the parameters, these corrections are substituted into the original equations of condition and the residuals calculated. For example, if a particular equation of condition is

$$\alpha_{i1}\varepsilon_1 + \quad \dots \quad + \alpha_{iN}\varepsilon_N = \Delta p^i$$

(where we have written α_{ij} for $\partial p^i/\partial e_j$)

then the residual we require from the equation is

$$[25] \qquad \nu_i = \alpha_{i1}\varepsilon_1 + \dots + \alpha_{iN}\varepsilon_N - \Delta p^i$$

where $\varepsilon_1, \dots, \varepsilon_N$ are the values of the unknowns (corrections to the parameters) obtained by solving the normal equations.

We then form the standard errors in the residuals E from

$$[26] \qquad E = \{\sum_i \nu_i^2/(m-n)\}$$

where m is the number of equations of condition and n is the number of free parameters.

The standard error of the correction ε_j to the parameter e_j, and hence to the parameter itself, is then

$$\sigma_j = E\,\sqrt{S_{jj}} \qquad [27]$$

where S_{jj} is the j^{th} diagonal component of the inverse of the normal matrix.

If we require an error estimate with a different level of probability, we multiply the standard error by a factor k such that $p = \mathrm{erf}(k/\sqrt{2})$. The <u>probable error</u> corresponds to a probability $p = \frac{1}{2}$ and we find that $k = 0.6745$ (cf. Brouwer and Clemence (1961) p.226). Thus the 'true' value of the parameter e_j is as likely to lie within the range $e_j - 0.6745\sigma_j$ to $e_j + 0.6745\sigma_j$ as it is to lie outside it. We present below a table of probabilities as a function of multiples of the standard error.

k	p x 100%
0.5	38.3%
0.6745	50.0%
1.0	68.3%
2.0	95.4%
3.0	99.73%
4.0	99.9937%
5.0	99.999943%

5.5.2 CORRELATIONS BETWEEN PARAMETERS

It is implicitly assumed in the least-squares correction process that the free parameters are independent of one another. That is to say, small corrections can be applied to any of the parameters independently and there are no relationships of the form

$$\varepsilon_i + \alpha\varepsilon_j = \beta \qquad [28]$$

where ε_i, ε_j are the corrections to parameters e_i, e_j and α, β are constants.

In practise, we often find such relationships and these may be explained in terms of the physics of the problem. Consider, for example, the case where the major semi axis of a satellite orbit and the mass of the primary are both free parameters in, say, a numerical integration model. These are related, together with the mean motion of the satellite, by Kepler's third law

$$n^2a^3 = k^2M \qquad [29]$$

where k^2 is the gravitational constant and we neglect the mass of the satellite itself. If we rewrite this equation in terms of small changes then we have

$$2\frac{\Delta n}{n} + 3\frac{\Delta a}{a} = \frac{\Delta M}{M}. \qquad [30]$$

Now the mean motion of the satellite will be well-determined by a series of observations spanning several decades, an interval equivalent to hundreds of orbital periods of the satellite. Thus the value of Δn is far better determined than either Δa (which depends, among other things, upon the calibration of the micrometer screw used to make separation measures) or ΔM. This is precisely the situation which gives rise to correlations : a linear combination of two unknowns is better-determined than either of the individual unknowns. Δa and ΔM are not independent since the solution will adjust them to minimise the change to Δn whose value is, as it were, a 'known constant' of the system. Thus we may expect to find correlations between the mass of the primary and the major semi axes of each of the satellites.

We may detect and quantify the correlations in a least-squares correction process by inspection of the normal matrix and its inverse. We define the correlation coefficient between ε_i and ε_j to be

[31] $$c_{ij} = S_{ij} / \sqrt{(S_{ii} \times S_{jj})}$$

where S denotes the inverse of the normal matrix. The correlation coefficient lies between -1 and +1 and it is normally close to zero i.e. the off-diagonal components of S are much smaller than those in the diagonal. However, when c_{ij} is nearly ±1 then ε_i and ε_j are closely correlated and it may prove difficult to obtain a solution for both of them simultaneously. We may show that the quantity

[32] $$\kappa_{ij} = a_{ij} / \sqrt{(a_{ii} \times a_{jj})}$$

also approaches ±1 when a pair of parameters are correlated. Suppose we have a set of equations of condition

$$\alpha_{i1}\varepsilon_1 + \alpha_{i2}\varepsilon_2 + \ldots + \alpha_{iN}\varepsilon_N = \beta_i$$

for i =1 to M. Suppose also that ε_1 and ε_2 are linearly dependent so that for some constant γ, $\alpha_{i2} = \gamma\alpha_{i1}$(or very nearly) for all equations of condition. Consider the components a_{11}, a_{12} and a_{22} in the normal matrix. We see that

[33] $$a_{12} = \gamma a_{11}$$

[34] $$a_{22} = \gamma^2 a_{11}$$

hence

[35] $$\kappa_{12} = \gamma a_{11}/(a_{11} \times \gamma^2 a_{11}) = \text{sgn } a_{11} = \pm 1.$$

It is interesting to note that γ may be found from

[36] $$\gamma = a_{12}/a_{11} = a_{22}/a_{12}.$$

Brouwer and Clemence (1961, p.231) present a method whereby the normal equations are re-written in order to solve them for the combination $\varepsilon_1 + \gamma\varepsilon_2$. Alternatively, we may choose to remove one of the parameters from the system, generally because we have a better determination of it from another source. This may be achieved quite easily by operating upon the normal equations. Suppose we wish to hold the value of e_k constant.

This means that we must force the normal equations to yield $\varepsilon_k = 0$. We set all the components of row k and column k of the normal matrix to zero <u>except</u> for the diagonal component, which we set to unity.

[37] $$a_{ik} = a_{ki} = 0 \qquad \text{for } i=1 \text{ to } N \text{ except } i=k$$

[38] $$a_{kk} = 1$$

and we set the k^{th} component of the right-hand-side vector $\underline{b}$ to zero.

[39] $$b_k = 0$$

This does not alter the normal matrix with respect to the other free parameters.

5.5.3 CONVERGENCE OF THE ITERATIVE PROCESS

We may regard the iterative correction process upon the parameters of our model as complete when certain criteria are met. Clearly, we wish successive sets of parameters to give an increasingly close fit of the model to the observational data. We may quantify this in terms of the root-mean-square (RMS) residuals of the data. The RMS residuals should decrease in size, settling to a constant value as convergence is attained. The number of data included in the RMS residuals must not decrease : as the closeness of the fit improves, we expect more observations to be included within the chosen rejection limit.

During each least-squares correction cycle, we may also consider the size of the corrections to be applied to the parameters. Corrections should decrease in size in successive iterations. For each of the free parameters e_j, we compare the correction ε_j with the standard error σ_j. When the correction is much smaller than the standard error, we may regard the correction as negligible. Thus when, say, $|\varepsilon_j|/\sigma_j << 0.01$ for all of the free parameters, the correction process is complete. Further iterations will not increase the accuracy with which the parameters can be determined.

5.6 RESULTS (2) : SOLUTION FOR POSITION AND VELOCITY

The first trial involved fitting the numerical integration to visual observations of Titan, Hyperion and Iapetus by solving for the position and velocity vectors of the three satellites and the masses and J parameters. The initial values of the masses and J parameters were those of Sinclair and Taylor (1985) while the initial values of the position and velocity were those obtained by fitting the integration to the analytical theories as described in section 5.4

The initial parameters gave quite a good fit to the observations.

Pair	RMS	Obsns	(%)
Saturn-Titan	1".069	299/321	93
Saturn-Hyperion	1".958	905/938	96
Saturn-Iapetus	0".853	315/321	96
Titan-Hyperion	1".576	688/766	90
Titan-Iapetus	0".696	820/832	99

All observations with an O-C residual of less than 5".0 are included. The fit for Hyperion is not as good as that for Titan and Iapetus because its initial parameters were not determined by iterative fitting to the theory. Nevertheless, 97% of all data are included in the table above.

When the normal equations were constructed from the residuals, a number of significant correlations were found. Among these, the mass of Saturn and its J_2 form factor were both strongly correlated with the initial position and velocity components of the satellites. In addition, the corrections to be applied to J_4 and the mass of Iapetus were larger than any physically reasonable value : the corrected mass of Iapetus would have been negative while the corrected value of J_4 would have exceeded J_2. Accordingly, it was decided to fix the masses and J's and a solution was made for the position and velocity components of the satellites only.

There were again a number of correlations. The x and y position components correlated strongly with the x and y velocity components for each satellite.

Titan x_o with y'_o

y_o with x'_o

Hyperion x_o with y_o

x_o with x'_o

x_o with y'_o

y_o with y'_o

x'_o with y'_o

Iapetus y_o with x'_o

The numerical integration was repeated using the corrected parameters. The O-C residuals resulting from the comparison of this integration with the observations were as follows.

Pair	RMS	Obsns	(%)
Saturn-Titan	2".413	36/321	11
Saturn-Hyperion	2".430	123/938	13
Saturn-Iapetus	0".827	315/321	98
Titan-Hyperion	2".468	351/706	50
Titan-Iapetus	2".397	594/832	71

The rejection limit, as before, is 5".0. The number of observations falling within this limit is much reduced and the RMS residuals are larger, with the exception of Saturn-Iapetus data. Clearly, the least-squares correction process has failed : the corrections applied to the parameters of Titan and Hyperion contain serious errors which are due to the large correlations.

The correlations between initial position and velocity components may be explained by postulating that some function of the position and velocity of each satellite is better-determined by the observations than any of the six individual components. If we consider the motion of a satellite in a nearly circular orbit in the xy-plane (this is approximately true for all the satellites in the numerical integration) then we may write

$$x = a\cos(nt + \varepsilon) + O(e)$$
$$y = a\sin(nt + \varepsilon) + O(e)$$

and

$$x' = -na\sin(nt + \varepsilon) + O(e)$$
$$y' = +na\cos(nt + \varepsilon) + O(e).$$

Neglecting the eccentricity of the orbit we may write

$$x' \simeq -n\,y$$
$$y' \simeq +n\,x.$$

This will be true at each point in the orbit and so the initial position and velocity components will be related thus

$$x'_o \simeq -n\,y_o$$
$$y'_o \simeq +n\,x_o.$$

It is significant that the strongest correlations in the least-squares fitting process are between x_o and y'_o and between y_o and x'_o. This leads us to suspect that relationships such as those above are indeed affecting the initial position and velocity components.

It would be more instructive to employ as the unknowns a set of parameters with more direct physical significance than position and velocity components. Therefore we turn our attention to the osculating elements of the orbits.

5.7 USE OF OSCULATING ELEMENTS AS AUXILIARY PARAMETERS

The osculating elements of the orbit of a satellite at any instant are the elements of the elliptic two-body orbit in which the unperturbed position and velocity of the satellite are the same as the actual position and velocity of the satellite at the epoch. There exist standard formulae for calculating osculating elements from a given set of position and velocity components, and vice versa. The most lucid exposition is that of Herrick (1971, chapter 4).

We wish to replace the position and velocity components of each satellite by the osculating elements as 18 of the parameters to be determined by comparison of the integration with the observations. To do this, we must change the iterative scheme, which now becomes :

1. Execute the numerical integration with the current set of position and velocity parameters and compare this with the observations to obtain the O-C residuals and equations of condition in terms of cor-

rections to the position and velocity parameters. This is exactly the same as before.

2. Convert the position and velocity components of each satellite at the zero epoch of the integration into osculating elements, and transform the equations of condition so that they express the O-C residuals in terms of corrections to the orbital elements. This transformation is carried out by use of partial derivatives of the position and velocity components with respect to the osculating elements. If we write the three position and velocity components of each satellite as the 18 components of a state vector ξ and the osculating elements as the 18 components of a vector $\underline{e}$ then

$$[40] \quad \Delta\xi_i = \sum_{j=1}^{18} \frac{\partial\xi_i}{\partial e_j} \Delta e_j .$$

This allows us to replace each $\Delta\xi_i$ in the equations of condition by a linear combination of the Δe_j's. Each of the equations is transformed from

$$[41] \quad \frac{\partial p}{\partial\xi_1} \Delta\xi_1 + \quad \ldots \quad + \frac{\partial p}{\partial\xi_{18}} \Delta\xi_{18} + \quad \frac{\partial p}{\partial J_2} \Delta J_2 + \ldots = \Delta p$$

to

$$[42] \quad \frac{\partial p}{\partial e_1} \Delta e_1 + \quad \ldots \quad + \frac{\partial p}{\partial e_{18}} \Delta e_{18} + \quad \frac{\partial p}{\partial J_2} \Delta J_2 + \ldots = \Delta p.$$

We note that the derivatives with respect to the masses and J factors are unchanged by the transformation since these parameters are the same whether we use position and velocity or osculating elements. We may write the general rule for the transformation of the coefficients of the equations as

[43] $$\frac{\partial p}{\partial e_k} = \sum_{j=1}^{18} \frac{\partial p}{\partial \xi_j} \frac{\partial \xi_j}{\partial e_k}.$$

The transformation coefficients $\partial\xi_j/\partial e_k$ are mostly zero : we need only calculate the derivatives of the position and velocity components of each satellite with respect to its own osculating elements. The required derivatives may be readily obtained from the classical formulae of two-body motion. Herrick (1972, section 15B) gives an account of the calculation of these derivatives using a method based upon the radial, transverse and binormal vectors of the orbit of the satellite.

3. Construct a set of normal equations from the new equations of condition, applying a suitable rejection limit (in this case, 2".0) to the O-C residuals as before.

4. Solve the normal equations to determine the corrections to the osculating elements (and the masses and J parameters if required).

5. Apply the corrections to the elements and convert them to instantaneous position and velocity components for each satellite using standard two-body elliptic formulae. The position and velocity vec-

tors thus obtained are the current parameters for a new iterative cycle beginning again at (1)

The osculating elements used in this scheme are referred to the coordinate system of the integration. Thus the inclination of each satellite's orbit is with respect to the equator plane of Saturn. Likewise, the node, apse and mean longitude are measured in the equator plane of Saturn from the x-axis of the integration coordinate system to the ascending node of the orbit upon the equator plane, and thence in the orbit plane.

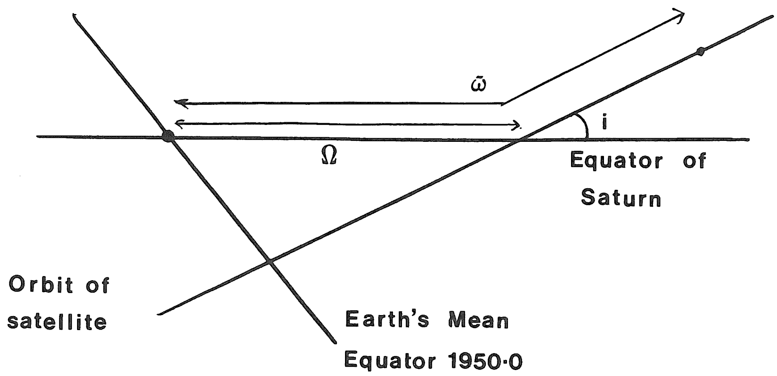


Figure 12. Angular osculating elements

5.8 RESULTS : (3) SOLUTION FOR OSCULATING ELEMENTS

A number of trials were carried out using various sets of free parameters. They are represented in diagram form below.

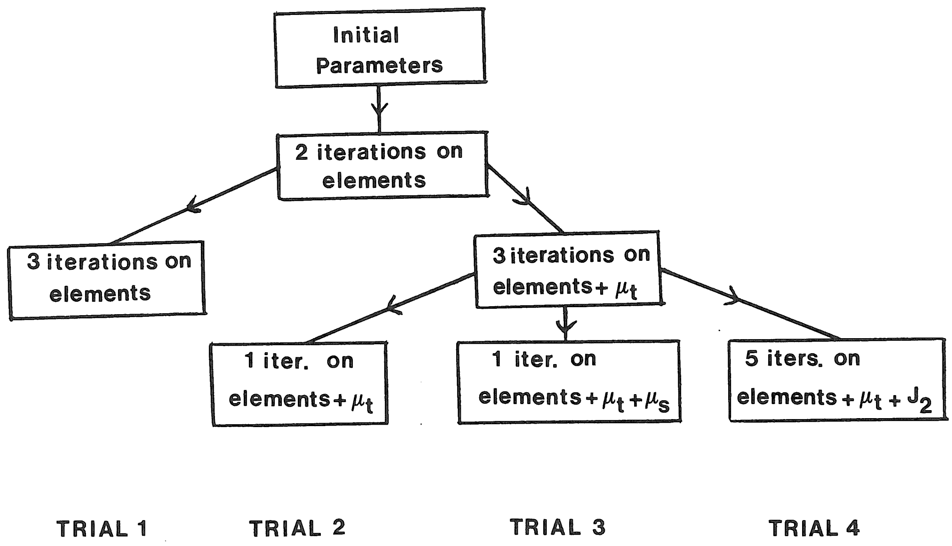


Figure 13. Solutions for osculating elements

In this context, 'element' signifies the set of 18 parameters comprising the six osculating elements for each of the three satellites. μ_T and μ_S are the mass ratios Titan/Saturn and Saturn/Sun respectively and J_2 is the dynamical form-factor of Saturn.

As an example, consider trial 3. Starting with the initial parameter set (obtained by fitting the integration to the analytic theories) we carry out the correction process upon the elements twice, then include μ_T for three correction cycles and finally include μ_s in one cycle. Thus trial 3 consists of six iterations incorporating an increasing number of free parameters. This approach allows us to monitor the behaviour of the least-squares process as new parameters are added : when convergence is attained with a given parameter set, we may increase the set to include further parameters of physical interest.

Trial 1 Five iterations on the osculating elements alone

The correction process converged quite rapidly and the RMS residuals after the 5th iteration are given below. The rejection limit for individual O-C residuals is 2".0. Results are given for both position angle and separation measures. Note that the position angle residuals are in fact $s\Delta p$ and thus they may be compared directly with the separation residuals.

Objects observed	Datum	Observations	RMS	Mean
Saturn - Titan	P	160 of 176	0".460	-0".105
Saturn - Titan	S	137 of 145	0".509	0".157
Saturn - Iapetus	P	154 of 162	0".747	0".064
Saturn - Iapetus	S	153 of 159	0".716	0".462
Saturn - Hyperion	P	458 of 473	0".743	-0".167
Saturn - Hyperion	S	433 of 465	0".871	0".514
Titan - Iapetus	P	411 of 417	0".385	0".009
Titan - Iapetus	S	407 of 415	0".357	0".186
Titan - Hyperion	P	344 of 355	0".452	-0".046
Titan - Hyperion	S	340 of 351	0".530	0".107
Iapetus - Hyperion	P	13 of 13	0".352	0".140
Iapetus - Hyperion	S	13 of 13	0".245	0".055

A total of 3023 out of 3144 (96.2%) data fall within the rejection limit.

The RMS residuals compare well with those obtained by Sinclair and Taylor (1985) who fitted a numerical integration to photographic (astrometric) observations covering the period 1967 to 1982. They obtained

Titan - Hyperion 0".33

Titan - Iapetus 0".22.

These figures are to be compared with the values obtained by combining the sΔp and Δs residuals above, which yield

Titan - Hyperion 0".49

Titan - Iapetus 0".37.

The residuals from visual observations are larger than those from photographic data by about 50%. This can probably be explained by the lower accuracy of the visual observations i.e. the random errors of observation are larger for the older data.

We must recall that the solution has been obtained by fitting to Saturn - satellite data as well as inter-satellite data, while Sinclair and Taylor used only inter-satellite measures. The RMS residuals for measures relative to Saturn are significantly larger than those for inter-satellite measures. This is due in part to the observational errors introduced in estimating the centre of the disk of Saturn when making separation and position angle measures relative to the planet. Inclusion of Saturn - satellite measures may therefore be expected to degrade a solution for the parameters of any dynamical model.

It may be more realistic to compare the residuals of Sinclair and Taylor with those obtained by fitting the integration to inter-satellite measures alone. Of the 3144 observations included in this study, 1580 are relative to Saturn while 1564 are inter-satellite measures : by excluding the Saturn - satellite measures we reduce the data set by 50% and advance the date of the earliest observation by 10 years, reducing the time span by 20%. The question of whether to employ Saturn - satellite data is thus a matter of compromise. A weighting scheme is probably the optimum solution but this introduces problems of its own and is not included in this work. It is deferred to the section on suggestions for further work.

Trial 2 Determination of the mass of Titan

This trial consisted of six iterations. During the first two, only the osculating elements were allowed to vary and so these iterations are the same as those of trial 1. Then the mass ratio Titan/Saturn was included in the set of free parameters and a further four iterations were carried out. Convergence was obtained (in the sense described in section 5.7) and the RMS residuals at a rejection limit of 2".0 were

Objects observed	Datum	Observations	RMS	Mean
Saturn - Titan	P	160 of 176	0".459	-0".103
Saturn - Titan	S	137 of 145	0".507	0".159
Saturn - Iapetus	P	154 of 162	0".747	0".064
Saturn - Iapetus	S	153 of 159	0".715	0".463
Saturn - Hyperion	P	457 of 473	0".731	-0".176
Saturn - Hyperion	S	435 of 465	0".878	0".523
Titan - Iapetus	P	411 of 417	0".385	0".009
Titan - Iapetus	S	407 of 415	0".357	0".185
Titan - Hyperion	P	345 of 355	0".464	-0".040
Titan - Hyperion	S	340 of 351	0".532	0".117
Iapetus - Hyperion	P	13 of 13	0".334	0".103
Iapetus - Hyperion	S	13 of 13	0".251	0".073

The mass of Titan was determined to be

$$\mu_T = (2.36659 \pm 0.00027) \times 10^{-4}.$$

The final RMS residuals are almost exactly the same as those in trial 1. Three more observations fall within the 2".0 limit than in trial 1 though one Saturn-Hyperion measure is lost. Thus, statistically speaking, this solution is based upon a data set of the same size as that used in trial 1.

<u>Trial 3</u> Determination of the mass of Saturn

This trial proceeded in the same way as trial 2 except that in the 6th correction cycle, the mass ratio Saturn/Sun (μ_s) was also allowed to vary. The free parameters were thus the osculating elements of Titan, Hyperion and Iapetus, the mass ratio Titan/Saturn and the mass ratio Saturn/Sun.

Upon carrying out the least-squares correction process on this parameter set, a number of significant correlations were found. The most notable were those between the mass of Saturn and the major semi axes of each of the satellites (as explained in section 5.5.2). To illustrate the effect of such correlations upon the solution, the corrections calculated by the least-squares method were applied to the parameters and a numerical integration was executed with the new parameters. This was compared with the observations and a set of RMS residuals calculated. The rejection limit in the table below is 2".0.

Objects observed	Datum	Observations	RMS	Mean
Saturn - Titan	P	5 of 176	1".026	-0.514
Saturn - Titan	S	4 of 145	1".352	-0.321
Saturn - Hyperion	P	6 of 473	1".366	-0.395
Saturn - Hyperion	S	6 of 465	1".354	0.369
Saturn - Iapetus	P	No observations within the limit		
Saturn - Iapetus	S	No observations within the limit		
Titan - Iapetus	P	7 of 417	1".256	0.044
Titan - Iapetus	S	4 of 415	1".186	-0.398
Titan - Hyperion	P	8 of 355	1".022	0.085
Titan - Hyperion	S	5 of 351	1".307	-0.165

Virtually no observations fall within the limit, and of course the RMS residuals do not possess any statistical significance. Clearly, the correlations implicit in the normal equations cause the corrections to some of the parameters to be in error. Accordingly, the value of the mass-ratio Saturn/Sun was held fixed in the subsequent trial.

Trial 4 Determination of Saturn's J_2

This trial was carried out as follows, starting with the initial parameter set.

1. Two iterations solving for osculating elements only.

2. Three iterations solving for elements plus μ_T.

3. Five iterations solving for elements plus μ_T plus J_2.

Convergence was obtained rather slowly following the introduction of J_2 as a free parameter, since several other parameters (notably major semi axes) underwent further changes to accommodate the new corrected value of J_2. This is a consequence of the form of the oblateness perturbations where J_2 is factored by a_e^2/r^3 (where a_e is the equatorial radius of Saturn and r is the radius vector of the satellite). Since r is proportional to the major semi axis, this factor is effectively a_e^2/a^3. Thus the major semi axis appears in the oblateness forces at a rather high power and a change in its value will also affect the value of J_2. Thus we might expect correlations between J_2 and the major semi axes of the satellite, and these do appear in the least-squares scheme. However, they do not cause erroneous corrections to be made to the parameters concerned.

The RMS residuals obtained after the final iteration are as follows. The rejection limit on individual O-C's is 2".0.

Objects observed	Datum	Observations	RMS	Mean
Saturn - Titan	P	160 of 176	0".462	-0".101
Saturn - Titan	S	137 of 145	0".513	0".160
Saturn - Iapetus	P	154 of 162	0".750	0".059
Saturn - Iapetus	S	153 of 159	0".712	0".461
Saturn - Hyperion	P	457 of 473	0".732	-0".169
Saturn - Hyperion	S	435 of 465	0".876	0".521
Titan - Iapetus	P	411 of 417	0".383	0".006
Titan - Iapetus	S	407 of 415	0".355	0".188
Titan - Hyperion	P	344 of 355	0".454	-0".043
Titan - Hyperion	S	341 of 351	0".539	0".121
Iapetus - Hyperion	P	13 of 13	0".330	0".073
Iapetus - Hyperion	S	13 of 13	0".235	0".064

Again, these RMS residuals are almost identical to those obtained in trials 1 and 2, with the same number of data falling within the 2".0 limit. Statistically, therefore, the parameters are based upon effectively the same data set as in trials 1 and 2. The values obtained for μ_T and J_2 are

$$\mu_T = (2.36651 \pm 0.00028)\ 10^{-4}$$

$$J_2 = 0.01779 \pm 0.00043$$

using a value of 60000 km (4.0107×10^{-4} AU) for the equatorial radius of Saturn.

5.9 DISCUSSION OF RESULTS

Values of the mass of Titan

We present below the values determined for the mass of Titan in this work and in two other recent papers. Sinclair and Taylor's (1985) value was obtained by fitting a numerical integration to photographic observations over the period 1967 to 1982. Tyler et al (1981) derived their value from analysis of radio tracking of the Voyager 1 spacecraft. We also include values calculated by Message based upon comparison of his theory of the motion of Hyperion with Woltjer's opposition mean data. Value (a) is a weighted average of values determined from individual terms of the theory whilst value (b) is a least-squares solution.

Source	$\mu_T \times 10^{-4}$
Trial 2	2.36659 ± 0.00027
Trial 4	2.36651 ± 0.00028
Sinclair and Taylor (1985)	2.36777 ± 0.00055
Tyler et al (1981)	2.3664 ± 0.0008
Message (a)	2.3648 ± 0.0055
Message (b)	2.3677 ± 0.0004

There is good agreement between our values and that of Tyler et al., though (as with J_2) they do differ by several standard errors from Sinclair and Taylor's determination. It is probably not sufficient to invoke an argument based on scale errors in visual observations to explain this discrepancy. The system is most sensitive to the mass of Titan via its perturbations upon Hyperion and thus a theoretical analysis of the dependence of the system upon Titan's mass will inevitably involve the theory of the motion of Hyperion. We do not propose to perform such an analysis in this work, recalling that one of the principal reasons for adopting numerical integration as a model was to avoid the complications of the theory of the motion of Hyperion !

Values of J_2

The value of J_2 includes the gravitational effect of the rings, the secular effect of the four inner satellites and any error in the mass adopted for Rhea. For comparison, the equivalent value obtained by Sinclair and Taylor is

$$J_2 = 0.01675 \pm 0.00089.$$

They write this combined or 'lumped' value as $\bar{J}_2$ and relate it to the true value by

$$[44] \qquad \bar{J}_2 = J_2 + 0.000061 + 52.6\ \Delta\mu_R$$

where the constant represents the secular contribution of the satellites Mimas, Enceladus, Tethys and Dione and of the rings, and $\Delta\mu_R$ is the correction required to the mass of Rhea. They assume that $\Delta\mu_R$ is zero and thus the true value of J_2 is obtained by subtracting 0.000061 from the 'lumped' value derived directly from observations. Thus for the true values we have

This work	0.01773 ± 0.00043
Sinclair and Taylor (1985)	0.01669 ± 0.00089
IAU (1976) recommended	0.01645.

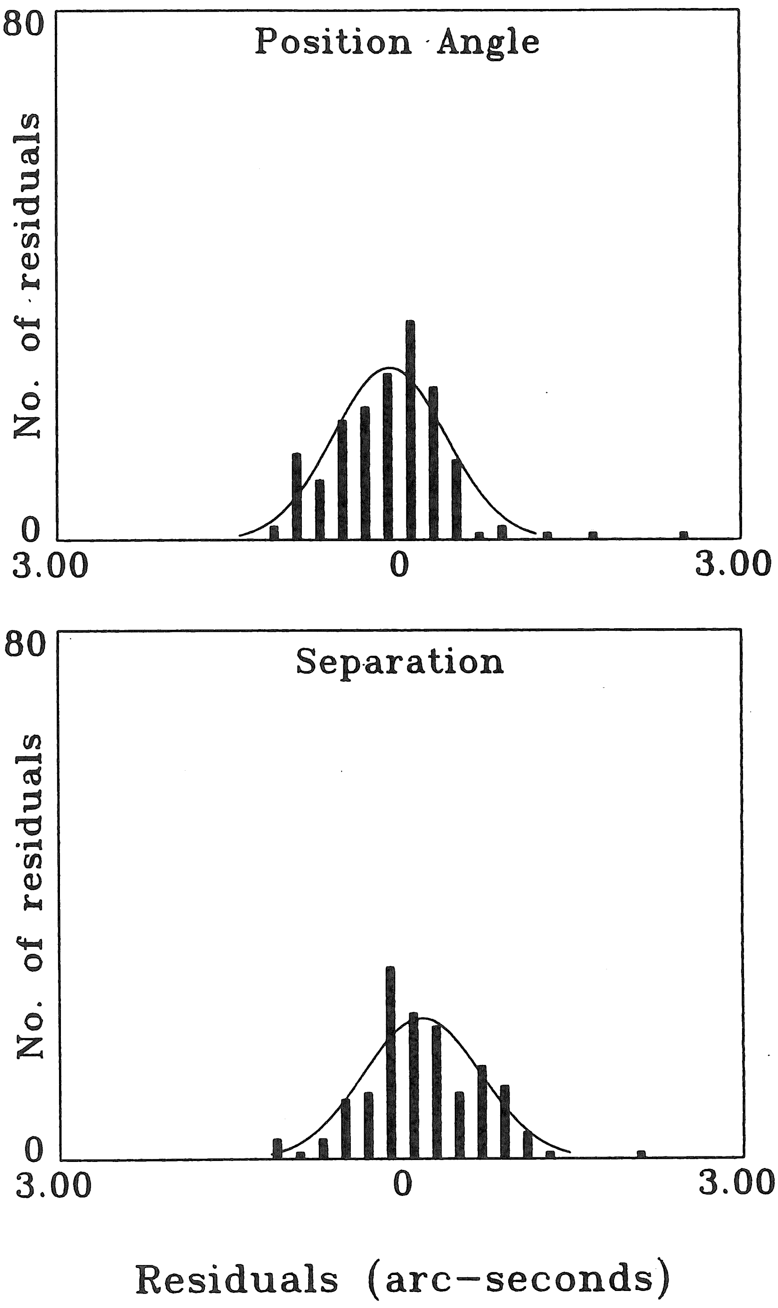
Our value is consistent with that of Sinclair and Taylor. They differ by 0.00104, rather more than one standard error of Sinclair and Taylor. As noted before, the oblateness perturbations are dependent upon the size of the satellite orbit in addition to J_2. Neglecting the eccentricity of the orbit, the principal factor in the second harmonic of the oblateness disturbing function includes J_2/a^3. A small change in the major semi axis will alter this factor quite considerably, hence changing the magnitude of the oblateness perturbations in the model and affecting the determination of J_2. This is the reason why the major semi axes are correlated with J_2. Visual observations often contain an error of scale which means that the values of the major semi axis of the orbit determined from such observations may include an error in the form of a multiplicative factor. For a particular observatory (i.e. a particular micrometer) this factor will be roughly the same for all the observed satellites and so the quantity J_2/a^3 will be in error by the same multiplicative factor for each of them. We may expect the observed value of J_2 to be a little too large or too small accordingly.

In the current work, the value of J_2 obtained from the integration is perhaps some 6% larger than most other determinations, which are closer to 0.165. This discrepancy is probably due to a scale error of the kind described.

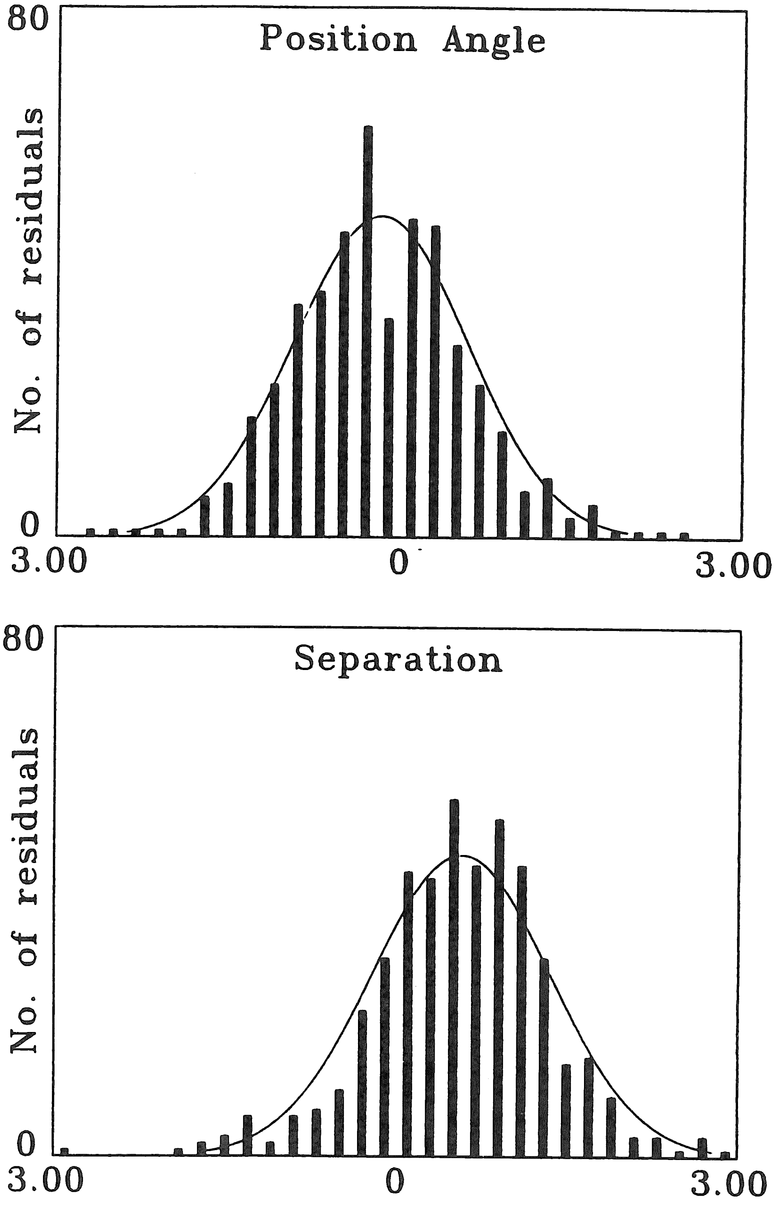
The standard error of our determination is half that of Sinclair and Taylor. We base our value on 50 years' data while Sinclair and Taylor have only 16 years of photographic observations. We note also that the two integrations have overall RMS residuals of similar size. The value of J_2 is determined from its secular effect on the nodes and inclinations of the satellite orbits and hence we expect better values from data which are spread over a greater time interval. Thus it is plausible that our standard error should be smaller than that of Sinclair and Taylor, though the discrepancy in the actual values of J_2 is disturbing and suggests that our value should be used with caution.

Distribution of the residuals

Saturn – Titan

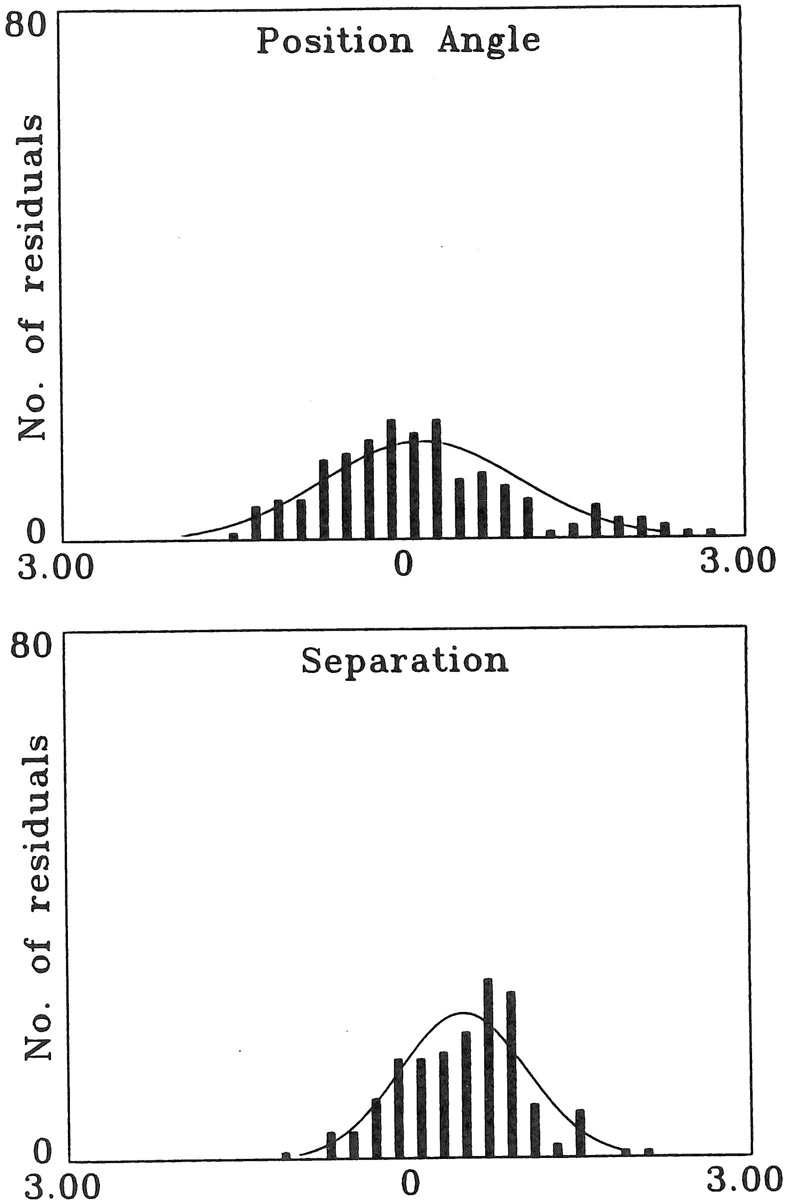


Saturn – Hyperion

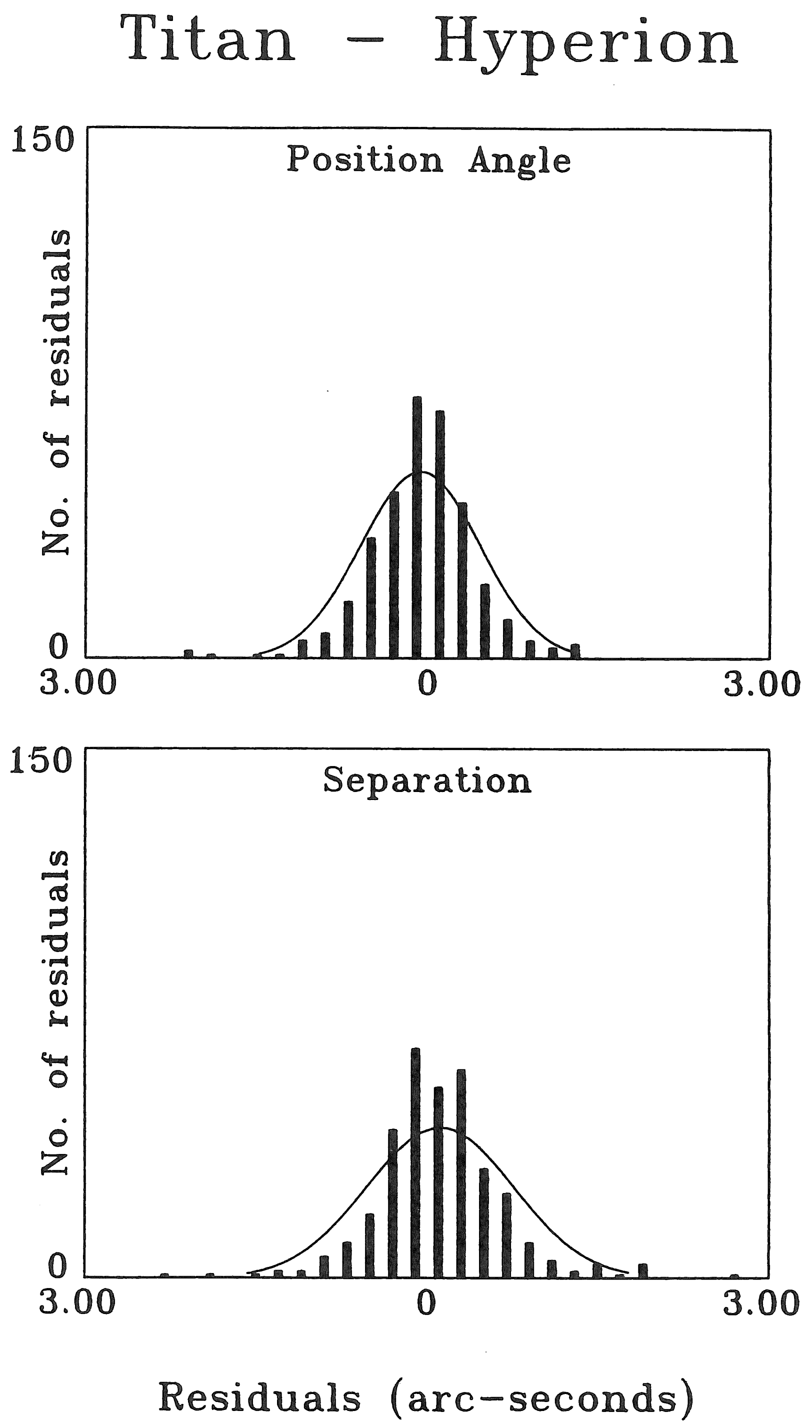


Residuals (arc–seconds)

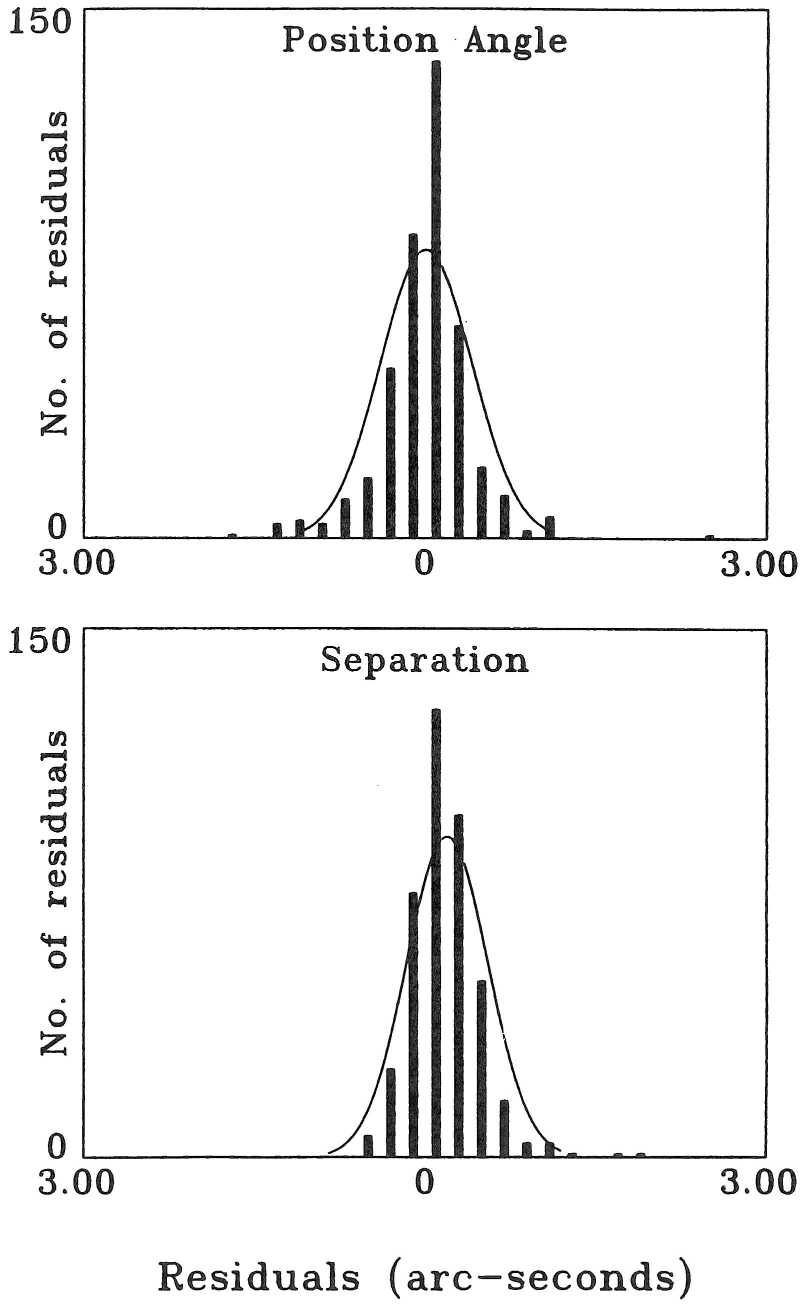
Saturn – Iapetus



Residuals (arc–seconds)


Titan - Hyperion
150
Position Angle
No. of residuals
0
3.00
0
3.00
150
Separation
No. of residuals
0
3.00
0
3.00
Residuals (arc-seconds)

Titan – Iapetus



In the accompanying diagrams we present a number of histograms of O-C residuals from Trial 3 to illustrate the distribution of residuals in all three of the successful (i.e. convergent) trials.

Most of the sets of residuals show a Gaussian distribution centred about zero. This is what we expect if the data are subject only to random errors and this is a basic premise of the least-squares correction process. It is interesting to note, however, that the mean residuals in separation measures of Saturn-Hyperion and Saturn-Iapetus are rather large : the distribution is still nearly Gaussian but with a significantly non-zero mean value. This suggests that these data contain systematic errors. The mean residuals are

Saturn - Titan	+0".160
Saturn - Hyperion	+0".522
Saturn - Iapetus	+0".462

where all the residuals less than 5".0 have been included.

We therefore seek a source of systematic errors which tends to increase the observed separation of Hyperion and Iapetus with respect to Saturn by about 0".5. A number of possibilities may be considered :

<u>The phase defect of the disk of Saturn as seen from the Earth.</u>

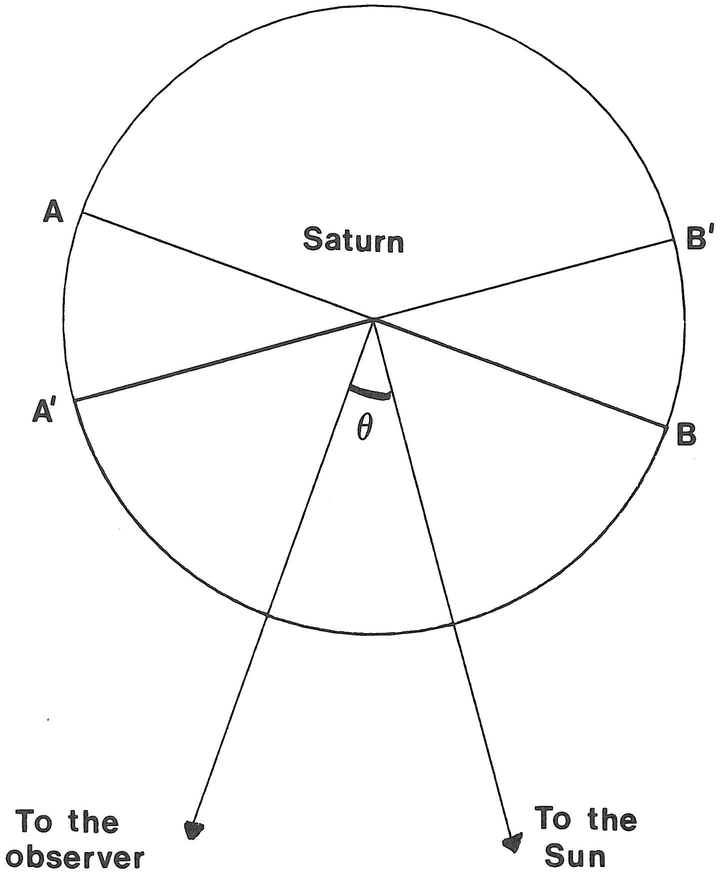


Figure 14. Phase defect of the disk of Saturn

Except at opposition, the illuminated area of Saturn presented to the Earth will not be perfectly circular. One limb will be in shadow (AA' in the accompanying figure) and the visible part of the planet is the segment B to A'. Thus the apparent diameter of the disk is reduced by factor $\sin^2\vartheta/2$ where ϑ is the angle at Saturn subtended by the Sun and the Earth. This has a maximum value of 0.002755 if we assume the Earth and Saturn to move in circular coplanar orbits. Thus the apparent diameter of Saturn ($19''.4$ at mean opposition distance) may be reduced by approximately $0''.05$ at most by the phase defect. The separation between the planet and a satellite is made with respect to the apparent centre of the planet's disk and this will differ from the centre of the true disk by exactly half the phase defect, 0".025. This is too small to satisfy our requirements, and moreover it may act to increase or decrease the measured separation depending on whether the satellite is one the same side of the planet as the phase defect or not. Both cases are equally likely and the net effect on a large number of observations will be zero.

The phase defect may manifest itself in another way which depends upon the reflective properties of the disk of the planet. When the planet is viewed from a direction other than the sub-Solar direction, the centre of illumination of the disk (that is, the point at which the light intensity is greatest) may not coincide with the sub-Solar point. The observer may identify the centre of the disk with the centre of illumination and hence introduce a further error similar to the phase defect. However, as in the case of geometrical phase defect, the average effect upon a large set of observations should be zero. Moreover, it ought to affect all satellites equally unless we invoke some magnitude dependency. I am grateful to Dr. Kaare Aksnes for this suggestion.

The figure of the satellite

Iapetus is known to have an uneven surface albedo : part of the satellite is dark while other parts are brighter. As in the case of Saturn's phase defect, the observer measures separation with respect to the centre of the illuminated part of the satellite and this may not coincide with the centre of mass, introducing an error. However, the apparent diameter of Iapetus at mean opposition distance is 0".23, too small to explain the systematic error under consideration. Furthermore, Hyperion is even smaller (0".03 at mean opposition) yet it displays the systematic error even more markedly than Iapetus.

Thickness of the micrometer wire

The wire employed in the micrometer of the Washington 26-inch refractor is stated (Wash. Obs. 1874, Appendix 1) to have a thickness of 0".251 in the field of view. Systematic positioning of the wire so that its edge, rather than its centre, coincided with the object under scrutiny may explain part of the large mean separation residuals. However, we can only invoke this argument for the positioning of the wire upon the centre of the disk of Saturn since the same micrometer was used to make the series of inter-satellite measures from 1892 which do not show large mean residuals.

Calibration of the micrometer scale

Newcomb gives the distance corresponding to one revolution of the micrometer screw of the Washington instrument as

$$9''.9480 \pm 0''.0015$$

based upon transit measurements (Wash. Obs. 1874, Appendix 1). The error coresponds to about ±0".15 in a measured separation of 1000". The largest separation measured in the satellite system of Saturn is some 600" and the error in this measure would be 0".09.

The error is proportional to the measured separation and so it will cause an error of scale in the entire satellite system. This effect is well known and it results in the major semi-axes of all of the satellites being in error by the same factor. This in turn leads to an erroneous determination of the mass of Saturn when these major semi-axes are used with Kepler's third law.

This cannot, however, be used to explain the large mean separation residuals because the mean residuals are not in proportion to the maximum separation between each satellite and the planet, as we would expect of they were due to multiplicative scale errors.

Unsteadiness of the atmosphere

Turbulence in the atmosphere prevents any celestial object from appearing as a sharply-defined disk or point source. A satellite whose apparent diameter would be 0".5 in the absence of an atmosphere will appear to the observer as a disk perhaps one arc-second across, depending upon local atmospheric conditions. Thus the observer cannot measure the position of the centre of the true disk but only the centre of a larger disk produced by the rapid motion of the object, a motion far faster than the response time of the human eye. This, however, will affect inter-satellite observations in the same way as Saturn-satellite observations and so it cannot be used to explain the anomalously large separation mean residuals for Saturn-satellite observations. I thank Hal Levison for suggesting this to account for residuals in micrometer observations.

Personal errors of observation

Each observer introduces some small systematic error into the data by virtue of his chosen observational technique. It may be significant that most of the Saturn-satellite observations used in this work were made by one observer (Hall). The main exception is the 1874 series of observations which were undertaken by Newcomb. These data, therefore, carry the stamp of Hall's personal error. This may have been a tendency to determine separation measures too large by about half an arc-second for the fainter satellites. Analysis of Hall's observations of the other faint satellites of Saturn during this period could provide further useful information to support this hypothesis, but that must be the subject of future work.

Final parameter sets

In addition to the masses and J factors determined (or adopted) in the three successful trials, the position and velocity vectors of the satellites were also determined at the epoch JED 2418800.5 and they are presented in appendix E.

5.10 CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The principal aim of the work in this chapter was to show that numerical integration can be used successfully in conjunction with visual micrometric observations to model the dynamics of a satellite system. As such, it is a natural extension of the work of Sinclair and Taylor who established the use of numerical integration with photographic astrometric observations. By its very nature, a numerical integration model must be compared to each observation individually : one cannot form opposition mean points as one might when using an analytical theory. This work is heavily dependent on computers, both to perform the integration itself and to carry out the least-squares comparison of the data with the integration. While integration has been widely used in modelling planetary and lunar dynamics since the late 1940s, its application to natural satellite dynamics is a new and developing field. This is due to a large extent to the fact that satellite theories do not generally need to give positions as accurately with respect to the primary in order to yield the same accuracy in the calculated values of the observed positions. It is also important to note that satellite theory has suffered a long period of relative neglect until the advent of space probes to Mars and the outer planets.

The absence of observational data over the period 1930 to 1967 is another problem in the study of the outer satellites of Saturn. With the death of G Struve, observations become scarce, particularly for Hyperion and Iapetus. Having fitted Sinclair's integration to data spanning 1874-1933 (this work) and 1967-1982 (Sinclair and Taylor 1985), the logical next step is to construct a model which is fitted to all the data, covering over a century. This is an ambitious plan and it may prove difficult. In principle, it is a straightforward calculation since both existing models provide partial derivatives with respect to the same parameters of the integration. Equations of condition may be combined to form a set of normal equations regardless of whether the equations of condition are generated by comparison with photographic observations or visual observations.

Among the problems, we may list

1. Significant disparities in the sizes of the random (and systematic) errors in the different types of observation. As we have noted, visual micrometric observations are subject to errors of scale which affect separation measures. Moreover, visual observations relative to Saturn may include systematic errors caused by the method used to make micrometer measurements.

   A system of weighting observations according to their type may be the solution to this problem, though it does not address the question of systematic errors.

2. The mean values of the separation residuals relative to Saturn tend to differ from zero. They should not do this if they were subject only to random errors. An investigation of the cause of this systematic error is necessary in order to give greater validity to the parameters determined in this chapter.

3. The distribution of the data over the 109 years of the proposed global numerical integration is very uneven. It is split into two sets of roughly equal size, about 3000 data in each. One set consists of quite accurate photographic observations over a period of 16 years, while the other is a collection of rather lower accuracy micrometric ob-

servations covering a period of 50 years. There is a 40-year gap between the two sets, where data are all but absent.

4. The resources required to perform such an integration may prove restrictive. Using Sinclair's integration program 'Titan' on an IBM 3083, the two 50-year runs would take a total of about 1200 seconds of CPU time, equivalent to many hours on a VAX machine. The least-squares analysis of the observations must also be added to this. Evidently, careful planning of the logistics of such a project would be essential.

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Washington Observations	(1874) 284 - 287
	(1875) 356 - 361
	(1876) 389 - 393
	(1877) 227 - 229
	(1878) 92 - 93
	(1879) 125 - 126
	(1880) 105 - 106
	(1881) 110 - 111
	(1882) 108
	(1883) 136 - 138
	(1884) 204 - 206

(1885) 177

(1886) 129 - 130

(1887) 103

(1888) 21

(1889) 95 - 96

(1890) 85 - 86

(1891) 45

(1892) 17

Observations made at USNO during the period 1893 to 1907 are contained in

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Publ. U.S. Naval Observatory, 17, part 3, 94 - 124

```
// Expansion of the Solar disturbing function in terms of the mutual
// inclination
//
//
// a = sin(I/2)
//
// e = eccentricity of satellite
// f = eccentricity of the Sun
//
// s = true anomaly of satellite
// t = true anomaly of the Sun
// u = apse angle of satellite
// v = apse angle of the Sun
// w = mean anomaly of satellite
// x = mean anomaly of the Sun
//
// define maximum order for expansion
//
WEIGHT(a)=0
WEIGHT(e)=1
WEIGHT(f)=1
MAXORDER=3
//
// First define cosS by spherical trigonometry
//
A=cos[s+u]cos[t+v]+(1-2a.2)sin[s+u]sin[t+v]
//
// The equation of the centre for the satellite
//
B=(2e-(1/4)e.3)sin[w] +((5/4)e.2-(11/24)e.4)sin[2w] +(13/12)e.3sin[3w]
B:=B+ (103/96)e.4sin[4w]
//
// The equation of the centre for the Sun
//
C=(2f-(1/4)f.3)sin[x] +((5/4)f.2-(11/24)f.4)sin[2x] +(13/12)f.3sin[3x]
C:=C+ (103/96)f.4sin[4x]
//
// The radius vector for the satellite (actually r/a)
//
D=1+(1/2)e.2 +(-e+(3/8)e.3)cos[w] +(-(1/2)e.2+(1/3)e.3)cos[2w]
D:=D- (3/8)e.3cos[3w] - (1/3)e.4cos[4w]
//
// The inverse radius vector of the Sun, in terms of the true anomaly
//
E=(1+f.2+f.4)(1+fcos[t])
```

```
//
// Now we substitute the equation of the centre to eliminate true anomalies
//
E:=HSUB(E,t,x,C,4)
//
A:=HSUB(A,s,w,B,4)
A:=HSUB(A,t,x,C,4)
//
// and multiply everything together
//
G:=DDEEE(-1+3AA)/2
PAGE
TEXT: Solar disturbing function:
PRINT[G]
STOP
END
```

APPENDIX B. FORMULAE FROM SPHERICAL TRIGONOMETRY

In several sections of this thesis, notably chapters 2 and 4, we seek differential relationships between parts of a spherical triangle. In general, we start with a set of formulae such as

$$\begin{aligned} \cos\vartheta &= \alpha \\ \sin\vartheta\cos\phi &= \beta \\ \sin\vartheta\sin\phi &= \gamma \end{aligned}$$

where ϑ, ϕ are the dependent variables and α, β, γ are functions of the independent variables, say w, x, y et cetera. We seek derivatives such as $\partial\vartheta/\partial w$, $\partial\phi/\partial w$. We may immediately write

$$-\sin\vartheta\, d\vartheta = d\alpha$$

and

$$\begin{aligned} \cos\vartheta\cos\phi\, d\vartheta - \sin\vartheta\sin\phi\, d\phi &= d\beta \\ \cos\vartheta\sin\phi\, d\vartheta + \sin\vartheta\cos\phi\, d\phi &= d\gamma \end{aligned}$$

whence

$$\begin{aligned} \cos\vartheta\, d\vartheta &= \cos\phi\, d\beta + \sin\phi\, d\gamma \\ \sin\vartheta\, d\phi &= -\sin\phi\, d\beta + \cos\phi\, d\gamma. \end{aligned}$$

Thus

$$\sin\vartheta \frac{d\vartheta}{dw} = -\frac{d\alpha}{dw}$$

$$\cos\vartheta \frac{d\vartheta}{dw} = \cos\phi \frac{d\beta}{dw} + \sin\phi \frac{d\gamma}{dw}$$

$$\sin\vartheta \frac{d\phi}{dw} = -\sin\phi \frac{d\beta}{dw} + \cos\phi \frac{d\gamma}{dw}.$$

As an example, we shall calculate the partial derivatives of Chapter 2, equation [21]. We may write the following formulae (cf. Explanatory Supplement to the Astronomical Ephemeris (1961) page 472).

(a) $\sin i \sin(\Omega - \Omega_s) = \sin\eta \sin(\Theta - \Omega_s)$

(b) $\sin i \cos(\Omega - \Omega_s) = \sin i_s \cos\eta + \cos i_s \sin\eta \cos(\Theta - \Omega_s)$

(c) $\cos i = \cos i_s \cos\eta - \sin i_s \sin\eta \cos(\Theta - \Omega_s)$

(d) $\sin i \sin\vartheta = \sin i_s \sin(\Theta - \Omega_s)$

(e) $\sin i \sin\vartheta = \cos i_s \sin\eta + \sin i_s \cos\eta \cos(\Theta - \Omega_s)$

From (c) we derive

$$-\sin i \, di/d\eta = -\cos i_s \sin\eta - \sin i_s \sin\eta\cos(\Theta - \Omega_s)$$
$$= -\sin i \cos\vartheta.$$

Thus $di/d\eta = \cos\vartheta$.

Also,

$$- \sin i \, di/d\Theta = \sin i_s \sin \eta \sin(\Theta - \Omega_s)$$
$$= \sin \eta \sin i \sin \vartheta.$$

Thus $\quad di/d\Theta = - \sin \vartheta \sin \eta.$

We introduce three further formulae.

(f) $\quad \sin \vartheta \cos \eta = \sin(\Theta - \Omega_s) \cos(\Omega - \Omega_s) - \cos(\Theta - \Omega_s) \sin(\Omega - \Omega_s) \cos i_s$

(g) $\quad \sin \vartheta \sin \eta = \sin i_s \sin(\Omega - \Omega_s)$

(h) $\quad \cos \vartheta = \cos(\Theta - \Omega_s) \cos(\Omega - \Omega_s) + \sin(\Theta - \Omega_s) \sin(\Omega - \Omega_s) \cos i_s$

and from (a) and (b) we obtain

$$\sin i \frac{d\Omega}{d\eta} = - \sin(\Omega - \Omega_s) \{-\sin i_s \sin \eta + \cos i_s \cos \eta \cos(\Theta - \Omega_s)\}$$
$$+ \cos(\Omega - \Omega_s) \{\cos \eta \sin(\Theta - \Omega_s)\}$$

$$= \sin \vartheta$$

and

$$\sin i \frac{d\Omega}{d\Theta} = - \sin(\Omega - \Omega_s) \{- \cos i_s \sin \eta \cos(\Theta - \Omega_s)\}$$

$$+ \cos(\Omega - \Omega_s) \{\sin \eta \cos(\Theta - \Omega_s)\}$$

$$= \sin \eta \sin \vartheta.$$

From (d) and (e) we have

$$\sin i \frac{d\vartheta}{d\eta} = - \sin \vartheta \{\cos i_s \cos \eta - \sin i_s \sin \eta \sin(\Theta - \Omega_s)\}$$

$$= - \sin \vartheta \cos i$$

and

$$\sin i \frac{d\vartheta}{d\Theta} = - \sin \vartheta \{- \sin i_s \cos \eta \sin(\Theta - \Omega_s)\}$$

$$+ \sin \vartheta \{\sin i_s \cos(\Theta - \Omega_s)\}$$

$$= \sin i_s \sin(\Omega - \Omega_s).$$

Thus we have the derivatives quoted in Chapter 2.

APPENDIX C. INDIRECT TERMS IN SOLAR AND SATELLITE PERTURBATIONS

In the Solar disturbing function of Chapter 2, and the contributions to the force model of Chapter 5 by the Sun and by mutual satellite perturbations, there appear terms known as <u>indirect terms</u> in addition to the direct inverse-square term. The acceleration of a satellite due to the action of the Sun is thus

$$\underline{a}_{is} = G M_s \left\{ \frac{\underline{r}_s - \underline{r}_i}{r_{is}^3} - \frac{\underline{r}_s}{r_s^3} \right\}$$

where G = the gravitational constant

M_s = mass of the Sun

$\underline{r}_s$ = Saturnicentric position vector of the Sun

$\underline{r}_i$ = Saturnicentric position vector of the i^{th} satellite

$r_{is} = | \underline{r}_s - \underline{r}_i |$

$r_s = | \underline{r}_s |$.

The first term is the direct gravitational attraction while the second term may be interpreted as the acceleration of the Sun relative to Saturn. It arises because the coordinate system is referred to the centre of Saturn (i.e. the centre of mass of Saturn, not of the entire system) and is thus not inertial.

Consider the equations of motion of the system consisting of Saturn, its satellites and the Sun, referred to an inertial frame. Then

(a) $$M_o \ddot{\underline{R}}_o = - GM_oM_s \left(\frac{\underline{R}_o - \underline{R}_s}{R_{os}^3} \right) - GM_om_i \left(\frac{\underline{R}_o - \underline{R}_i}{R_{oi}^3} \right)$$

where M_o = mass of Saturn

M_s = mass of the Sun

m_i = mass of the i^{th} satellite

$\underline{R}_o$ = position vector of Saturn

$\underline{R}_s$ = position vector of the Sun

$\underline{R}_i$ = position vector of the i^{th} satellite

and $$R_{os} = | \underline{R}_o - \underline{R}_s |$$

$$R_{oi} = | \underline{R}_o - \underline{R}_i | .$$

For the i^{th} satellite we have

(b) $$M_i \ddot{\underline{R}}_i = - GM_oM_i \left(\frac{\underline{R}_i - \underline{R}_o}{R_{io}^3} \right) - GM_sm_i \left(\frac{\underline{R}_s - \underline{R}_i}{R_{is}^3} \right) - \sum_{j}^{j \neq i} Gm_iM_j \left(\frac{\underline{R}_i - \underline{R}_j}{R_{ij}^3} \right)$$

where $\underline{R}_j$ = position vector of the j^{th} satellite.

We seek the acceleration of the i^{th} satellite relative to Saturn and thus we want

$$\ddot{\underline{r}}_i = \ddot{\underline{R}}_i - \ddot{\underline{R}}_o$$

where we use upper-case letters to denote position vectors in the inertial frame and lower-case letters to denote those in the Saturnicentric frame. Thus

$$\ddot{\underline{R}}_i = -G(M_o + m_i)\,\frac{\underline{R}_i - \underline{R}_o}{R_{io}^3} - G\,M_s\left(\frac{\underline{R}_i - \underline{R}_s}{R_{is}^3} + \frac{\underline{R}_s - \underline{R}_o}{R_{os}^3}\right) - \sum_{j}^{j\neq i} G\,M_j\left(\frac{\underline{R}_i - \underline{R}_j}{R_{ij}^3} + \frac{\underline{R}_j - \underline{R}_o}{R_{oj}^3}\right)$$

$$= -\; G(M_o + m_i)\,\frac{\underline{r}_i}{r_{io}^3} - G\,M_s\left(\frac{\underline{r}_i - \underline{r}_s}{r_{is}^3} + \frac{\underline{r}_s}{r_{os}^3}\right) - \sum_{j}^{j\neq i} G\,M_j\left(\frac{\underline{r}_i - \underline{r}_j}{r_{ij}^3} + \frac{\underline{r}_j}{r_{oj}^3}\right)$$

The origin of the indirect terms may be clearly seen from this derivation. In chapter 2, we also require the disturbing function for Solar perturbations. This is the function R_s such that

$$\text{grad } R_s = G\,M_s\left(\frac{\underline{r}_s - \underline{r}_i}{r_{is}^3} - \frac{\underline{r}_s}{r_s^3}\right).$$

The direct term is evidently GM/r_{is}. The indirect term in the acceleration does not contain the coordinates of the perturbed satellite and thus its contribution to the disturbing function may be written as

$$- G M_s \frac{\{x x_s + y y_s + z z_s\}}{r_s^3}$$

$$= - G M_s \frac{\underline{r} \cdot \underline{r}_s}{r_s^3}.$$

The Solar disturbing function is thus

$$R_s = G M_s \left(\frac{1}{r_{is}} - \frac{\underline{r} \cdot \underline{r}_s}{r_s^3} \right).$$

Recalling that $\underline{r} \cdot \underline{r}_s = r\ r_s \cos X$ where X is the angle subtended at the centre of Saturn by the position vectors $\underline{r}$ and $\underline{r}_s$, we may write this as

$$R_s = G M_s \left(\frac{1}{r_{is}} - \frac{r \cos X}{r_s^2} \right)$$

(cf. Brouwer and Clemence (1961) page 308, equation (1a))

APPENDIX D. EXPRESSIONS FOR OBLATENESS PERTURBATIONS

In chapter 5 we require the components of the force upon each satellite due to the oblateness of Saturn. We begin by considering the disturbing function for oblateness perturbations

(a) $$R_e = - GM/r \quad (a_e/r)^n J_n P_n(w)$$

where a_e = equatorial radius of Saturn

r = distance from the centre of Saturn

J_n = n^{th} harmonic coefficient of the potential field

w = z/r = latitude above the equatorial plane of Saturn

and $P_n(w)$ is a Legendre polynomial of degree n.

The acceleration upon a satellite at this point due to the oblateness of Saturn is

(b) $$\underline{a}_e = \nabla R_e .$$

Since Saturn is symmetrical about its equatorial plane, we may neglect all odd harmonics. The first term is thus the J_2 term. Consider the n^{th} harmonic term

(c) $$R_n = - (GM/r) J_n (a_e/r)^n P_n(w).$$

The components of the acceleration of the satellite due to this term are then

$$x_n = \partial R_n/\partial x$$
$$y_n = \partial R_n/\partial y$$
$$z_n = \partial R_n/\partial z.$$

Now we may write

$$\partial R_n/\partial x = + G\ M\ J_n\ (a_e/r)^n\ (x/r^3)\ \{(n+1)P_n(w) + w.P'_n(w)\}$$

$$= G\ M\ J_n\ (a_e/r)^n\ (x/r^3)\ P'_{n+1}(w).$$

The expression for $\partial R_n/y$ is exactly similar, writing y in place of x, while the expression for $\partial R_n/\partial z$ has an extra term

$$- G\ M\ J_n\ (a_e/r)^n\ (1/r^2)\ P'_n(w)$$

because w depends upon z directly, as well as indirectly via r. Thus we may write

$$\underline{a}_n = G\ M\ J_n(a_e/r)^n\ (1/r^2)\ \{P'_{n+1}(w)\ \ \underline{r} - P'_n(w)\ \ \underline{k}\}$$

where $\underline{r}$ is a unit vector in the radial direction

$\underline{k}$ is the unit vector in the z direction.

This is the form of the oblateness accelerations used by Sinclair and Taylor (1985).

APPENDIX E. FINAL SETS OF PARAMETERS FROM NUMERICAL INTEGRATION TRIALS

Epoch = Julian Ephemeris Date 2418800.5

	Trial 1	Trial 2	Trial 3
Titan			
$\underline{r}$	-0.0079438545	-0.0079440917	-0.0079429489
	0.0002251206	0.0002253203	0.0002302996
	-0.0000197461	-0.0000195983	-0.0000191867
$\dot{\underline{r}}$	-0.0001257187	-0.0001257718	-0.0001268372
	-0.0033045519	-0.0033044504	-0.0033048230
	0.0000183595	0.0000185320	0.0000186238
Hyperion			
$\underline{r}$	0.0058500907	0.0058503911	0.0058519167
	-0.0093650299	-0.0093636307	-0.0093600560
	0.0000713479	0.0000720276	0.0000710875
$\dot{\underline{r}}$	0.0021938372	0.0021938579	0.0021942391
	0.0013940284	0.0013944440	0.0013949571
	-0.0000290135	-0.0000288438	-0.0000288451
Iapetus			
$\underline{r}$	-0.0226958048	-0.0226958837	-0.0226951800
	0.0015958220	0.0015959033	0.0015987369
	-0.0056343356	-0.0056344345	-0.0056342563
$\dot{\underline{r}}$	-0.0001092249	-0.0001092299	-0.0001093554
	-0.0019102618	-0.0019102521	-0.0019102873
	0.0000870654	0.0000870693	0.0000870921
Masses and form-factors			
J_2	0.01675414 (*)	0.01675414 (*)	0.017788155
J_4	-0.00100000 (*)	-0.00100000 (*)	-0.00100000 (*)
μ_{Titan}	0.00023678 (*)	0.00023666	0.00023665
$\mu_{Iapetus}$	0.000003308 (*)	0.000003308 (*)	0.000003308 (*)
μ_{Saturn}	0.00028588 (*)	0.00028588 (*)	0.00028588 (*)

(*) not determined by fitting to observations : fixed value adopted.